1) $\underline{M} = \begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}$

$\det \underline{M} = 2 - -6 = 8$

$\underline{M}^{-1} = \frac{1}{8}\begin{pmatrix} 1 & -3 \\ 2 & 2 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{8} & -\frac{3}{8} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

Area multiplied by $\det \underline{M}$

$\therefore \quad 2 \times 8 = 16 \text{ units}^2$

---

2)

i) $\frac{1}{r+1} - \frac{1}{r+2} = \frac{(r+2)-(r+1)}{(r+1)(r+2)}$

$= \frac{r+2-r-1}{(r+1)(r+2)}$

$= \frac{1}{(r+1)(r+2)}$

---

ii) $\sum_{r=1}^{n} \frac{1}{(r+1)(r+2)}$

$= \sum_{r=1}^{n} \left(\frac{1}{r+1} - \frac{1}{r+2} \right)$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + + \frac{1}{n} + \frac{1}{n+1}$

$- \frac{1}{3} - \frac{1}{4} - - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$

$= \frac{1}{2} - \frac{1}{n+2}$

---

3) $\frac{1}{x+2} = 3x+4$

i) $1 = (3x+4)(x+2)$

$1 = 3x^2 + 4x + 6x + 8$

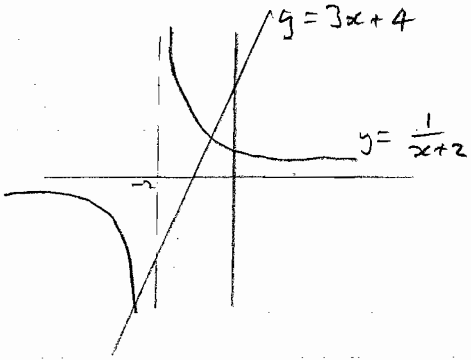
$0 = 3x^2 + 10x + 7$

$0 = (3x+7)(x+1)$

$\Rightarrow \quad x = -\frac{7}{3} \text{ or } x = -1$

---

3 ii) $\frac{1}{x+2} \leq 3x+4$



Graphs intersect at $x = -\frac{7}{3}$ and $x = -1$

Solution

$-\frac{7}{3} \leq x < -2$

or $x \geqslant -1$

---

4) $\sum_{r=1}^{n} r^2(r+2)$

$= \sum_{r=1}^{n} (r^3 + 2r^2)$

$$= \sum_{r=1}^{n} r^3 + 2\sum_{r=1}^{n} r^2$$

$$= \frac{1}{4}n^2(n+1)^2 + 2\left(\frac{1}{6}n(n+1)(2n+1)\right)$$

$$= \frac{n^2}{4}(n+1)^2 + \frac{n}{3}(n+1)(2n+1)$$

$$= \frac{n}{12}(n+1)\left[3n(n+1) + 4(2n+1)\right]$$

$$= \frac{n}{12}(n+1)\left[3n^2 + 3n + 8n + 4\right]$$

$$= \frac{n}{12}(n+1)(3n^2 + 11n + 4)$$

5)

$$x^3 + 2x^2 + x - 3 = 0$$

Roots α, β, γ

Find eqn with roots $\alpha+1,\ \beta+1,\ \gamma+1$

$\Sigma\alpha = -2$
$\Sigma\alpha\beta = 1$
$\Sigma\alpha\beta\gamma = 3$

$\alpha+1+\beta+1+\gamma+1 = \Sigma\alpha + 3 = 1$

$(\alpha+1)(\beta+1) + (\alpha+1)(\gamma+1) + (\beta+1)(\gamma+1)$

$= \alpha\beta + \alpha + \beta + 1 + \alpha\gamma + \alpha + \gamma + 1 + \beta\gamma + \gamma + \beta + 1$

$= \Sigma\alpha\beta + 2\Sigma\alpha + 3$

$= 1 - 4 + 3 = 0$

$(\alpha+1)(\beta+1)(\gamma+1)$

$= (\alpha\beta + \alpha + \beta + 1)(\gamma + 1)$

$= \alpha\beta\gamma + \alpha\gamma + \beta\gamma + \gamma + \alpha\beta + \alpha + \beta + 1$

$= \Sigma\alpha\beta\gamma + \Sigma\alpha\beta + \Sigma\alpha + 1$

$= 3 + 1 - 2 + 1 = 3$

Eqn is

$$x^3 - x^2 - 3 = 0$$

6) Prove $\displaystyle\sum_{r=1}^{n} r2^{r-1} = 1 + (n-1)2^n$

When $n=1$

$$1 \times 2^0 = 1 = 1 + 0 \times 2^1$$

$\therefore$ formula true for $n=1$

Assume true for some value $n=k$

then $\displaystyle\sum_{r=1}^{k} r2^{r-1} = 1 + (k-1)2^k$

but then

$$\sum_{r=1}^{k+1} r2^{r-1} = 1 + (k-1)2^k + (k+1)2^k$$

$$= 1 + (k-1+k+1)2^k$$

$$= 1 + 2k \times 2^k$$

$$= 1 + k \times 2^{k+1}$$

$$= 1 + ((k+1)-1)2^{k+1}$$

which is the same formula

6 cont) with $k+1$ replacing k.

$\therefore$ if true for $n=k$, formula is true for $n=k+1$.

Since true for $n=1$, by mathematical induction it is true for all values of n where n is a positive integer

Section B

7) $$y = \frac{(2x-3)(x+1)}{(x+4)(x-2)}$$

i) $y=0$ when $x=-1$ and when $x=\frac{3}{2}$

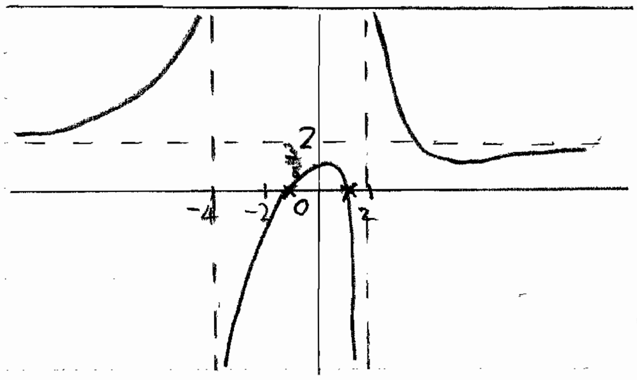
ii) Asymptotes $x=-4$, $x=2$, $y=2$

iii) A) When $x=100$, $y \approx 1.95$

B) When $x=-100$ $y \approx 2.05$

For large +ve x curve approaches horizontal asymptote from below

For large −ve x approaches from above.

iv) 

When $x=-4.1$ $y = \frac{- \times -}{- \times -} > 0$

when $x=-3.9$ $y = \frac{- \times -}{+ \times -} < 0$

when $x=1.9$ $y = \frac{+ \times +}{+ \times -} < 0$

when $x=2.1$ $y = \frac{+ \times +}{+ \times +} > 0$

v) Solve $\frac{(2x-3)(x+1)}{(x+4)(x-2)} \leq 2$

Solve $(2x-3)(x+1) = 2(x+4)(x-2)$

$2x^2 - 3x + 2x - 3 = 2(x^2+4x-2x-8)$

$\cancel{2x^2} - x - 3 = \cancel{2x^2} + 4x - 16$

$0 = 5x - 13$

$\Rightarrow x = \frac{13}{5} = 2.6$

Curve crosses $y=2$ at $x=2.6$

Solution

$-4 < x < 2$ or $x \geq 2.6$

8) $\alpha = 2 - j$ $\beta = -1 + 2j$

i) $\alpha + \beta = 1 + j$

$\alpha\beta = (2-j)(-1+2j)$

$= -2 + j + 4j - 2j^2$

$= -2 + 5j + 2 = 5j$

$\frac{\alpha}{\beta} = \frac{2-j}{-1+2j} \times \frac{-1-2j}{-1-2j}$

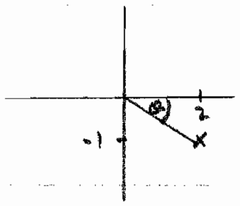
8 i) cont

$$\frac{\alpha}{\beta} = \frac{-2 + j - 4j + 2j^2}{1 + 4}$$

$$= -\frac{4}{5} - \frac{3}{5}j$$

ii)

$$|\alpha| = \sqrt{2^2 + (-1)^2} = \sqrt{5}$$

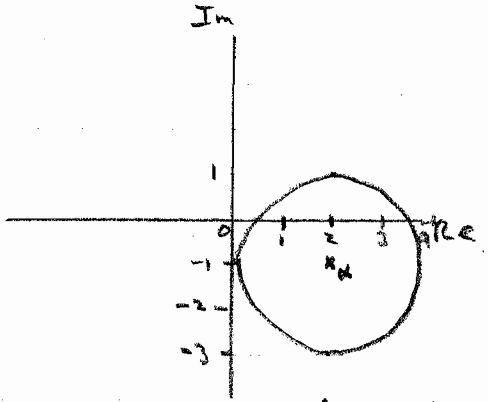


$\theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\theta = 0.464$ radians

Arg $\alpha = -0.464$ radians

iii)

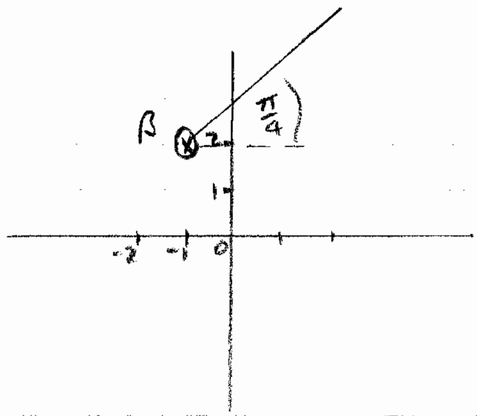


Circle centre $(2, -1)$

Radius $= 2$

iv)

$$\arg(z - \beta) = \frac{\pi}{4}$$



Half line argument $\frac{\pi}{4}$ from β, excluding point β

9) $\underline{M} = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}$

i) $\underline{M}^2 = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}\begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}$

$$\underline{M}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

ii) $\underline{M}^2$ is the identity matrix

If you reflect in the line twice, you get back where you started.

iii)

$$\begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$0.8x + 0.6y = x$ ①

$0.6x - 0.8y = y$ ②

From ① $0.6y = 0.2x$

$y = \frac{1}{3}x$

From ② $0.6x = 1.8y$

$\frac{1}{3}x = y$

Eqn of mirror line

$$y = \frac{1}{3}x$$

iv) P represents a rotation ant-clockwise about $(0,0)$ by $\theta°$ where $\theta = \cos^{-1} 0.8$

$\theta \approx 36.9°$

9v) $\underline{MP} = \begin{pmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{pmatrix}\begin{pmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

9vi) $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a reflection in the x-axis

$\therefore$ mirror line is $y = 0$

5) Alternative solution using substitution

Roots of $x^3 + 2x^2 + x - 3 = 0$

are α, β, γ

Find eqn with roots $\alpha+1, \beta+1, \gamma+1$

Let $w = x + 1$

then $x = w - 1$

$(\alpha+1), (\beta+1), (\gamma+1)$ will be

roots of $(w-1)^3 + 2(w-1)^2 + (w-1) - 3 = 0$

$(w^2 - 2w + 1)(w - 1) + 2(w^2 - 2w + 1)$

$+ (w - 1) - 3 = 0$

$w^3 - \cancel{2w^2} + \cancel{w} - w^2 + \cancel{2w} - \cancel{1}$

$+ \cancel{2w^2} - \cancel{4w} + \cancel{2} + \cancel{w} - \cancel{1} - 3 = 0$

$w^3 - w^2 - 3 = 0$

$\therefore \alpha+1, \beta+1, \gamma+1$

would be roots of

$x^3 - x^2 - 3 = 0$