MEI FP1 JUNE 2007



**1)** $\underline{M} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$

**i)** $\underline{M}^{-1} = \dfrac{1}{6--4}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$

$= \begin{pmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{-4}{10} & \frac{2}{10} \end{pmatrix}$

**1 ii)** Area multiplied by det M

$\therefore\ 2 \times 10 = 20$ units$^2$

**2)** $|z-(3-2j)| = 2$

**3)** $x^3 - 4 \equiv (x-1)(Ax^2+Bx+C)+D$

Coeff of $x^3$

$1 = A \qquad \Rightarrow A = 1$

Coeff of $x^2$

$0 = B - A$

$0 = B - 1 \qquad \Rightarrow B = 1$

Coeff of $x$

$0 = C - B$

$0 = C - 1 \qquad \Rightarrow C = 1$

Coeff of constant

$-4 = -C + D$

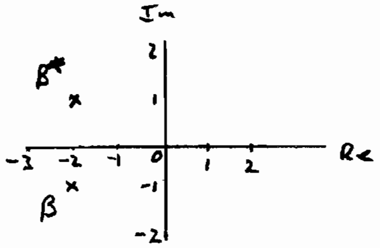
$-4 = -1 + D$

$-4 + 1 = D$

$\Rightarrow D = -3$

**4)** $\alpha = 1-2j, \quad \beta = -2-j$

**i)**




$\beta^* = -2+j$

**ii)**

$\alpha\beta = (1-2j)(-2-j)$

$= -2 + 4j - j + 2j^2$

$= -4 + 3j$

**iii)**

$\dfrac{\alpha+\beta}{\beta} = \dfrac{(1-2j)+(-2-j)}{(-2-j)}$

$= \dfrac{-1-3j}{-2-j}$

$= \dfrac{(-1-3j)}{(-2-j)} \times \dfrac{(-2+j)}{(-2+j)}$

$= \dfrac{2+6j-j-3j^2}{2^2+1^2}$

$= \dfrac{5+5j}{5} = 1+j$

**5)** $x^3 + 3x^2 - 7x + 1 = 0$

has roots $\alpha, \beta, \gamma$

Let $w = 3x$

$\Rightarrow x = \frac{w}{3}$

Subst in eqn

$\left(\frac{w}{3}\right)^3 + 3\left(\frac{w}{3}\right)^2 - 7\left(\frac{w}{3}\right) + 1 = 0$

5) cont

$$\frac{w^3}{27} + \frac{3w^2}{9} - \frac{7w}{3} + 1 = 0$$

$$\Rightarrow w^3 + 9w^2 - 63w + 27 = 0$$

Roots of this eqn are $3\alpha, 3\beta, 3\gamma$

6)

i) $$\frac{1}{r+2} - \frac{1}{r+3}$$

$$= \frac{(r+3)-(r+2)}{(r+2)(r+3)} = \frac{1}{(r+2)(r+3)}$$

ii) $$\sum_{r=1}^{50} \frac{1}{(r+2)(r+3)} = \sum_{r=1}^{50}\left(\frac{1}{r+2} - \frac{1}{r+3}\right)$$

$$= \begin{array}{rr} +\frac{1}{3} & -\cancel{\frac{1}{4}} \\ +\cancel{\frac{1}{4}} & -\cancel{\frac{1}{5}} \\ +\cancel{\frac{1}{5}} & -\cancel{\frac{1}{6}} \\ \vdots & \\ +\cancel{\frac{1}{51}} & -\cancel{\frac{1}{52}} \\ +\cancel{\frac{1}{52}} & -\frac{1}{53} \end{array}$$

$$= \frac{1}{3} - \frac{1}{53}$$

$$= \frac{53-3}{159} = \frac{50}{159}$$

7) Prove $$\sum_{r=1}^{n} 3^{r-1} = \frac{3^n - 1}{2}$$

When $n=1$

$$3^0 = \frac{3^1 - 1}{2}$$

$$1 = \frac{3-1}{2} \quad ✓$$

$\therefore$ formula true when $n=1$

Assume true for $n=k$

then $$\sum_{r=1}^{k} 3^{r-1} = \frac{3^k - 1}{2}$$

$$\Rightarrow \sum_{r=1}^{k+1} 3^{r-1} = \frac{3^k - 1}{2} + 3^k$$

$$= \frac{3^k - 1 + 2 \times 3^k}{2}$$

$$= \frac{3 \times 3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2}$$

This is the same formula with $k+1$ replacing $k$. $\therefore$ if formula is true for $n=k$, it is also true for $n=k+1$.

Since formula is true for $n=1$, by mathematical induction it is true for all positive integers $n$

8) $y = \dfrac{x^2-4}{(x-3)(x+1)(x-1)}$

i) $y = \dfrac{(x+2)(x-2)}{(x-3)(x+1)(x-1)}$

$y = 0$ when $x = \pm 2$

Cuts $x$-axis at $(2,0)$ and $(-2,0)$

When $x=0$, $y = \dfrac{-4}{(-3)(+1)(-1)} = -\dfrac{4}{3}$

Cuts $y$-axis at $\left(0, -\frac{4}{3}\right)$

ii) Vertical asymptotes
$x = 3$, $x = 1$, $x = -1$
Horizontal asymptote
$y = 0$

iii)
A) When $x = 100$

$y = \dfrac{100^2 - 4}{97 \times 101 \times 99} > 0$

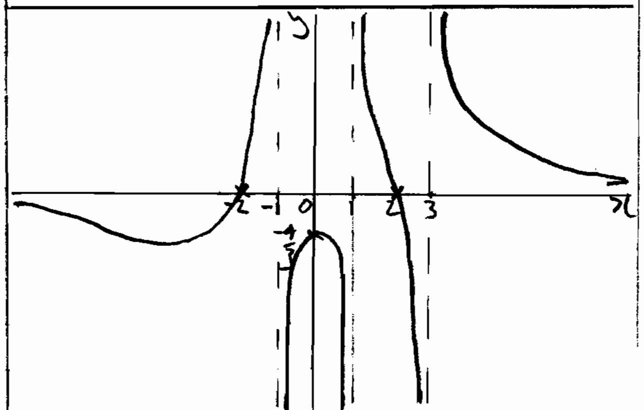
$\therefore y \to 0^+$ as $x \to \infty$

B) When $x = -100$

$y = \dfrac{(-100)^2 - 4}{(-103)(-99)(-101)} < 0$

$\therefore y \to 0^-$ as $x \to -\infty$

iv)




when $x = -1.1$ $y$ is $\dfrac{-}{- - -}$ +ve

$x = -0.9$ $y$ is $\dfrac{-}{- + -}$ −ve

$x = 0.9$ $y$ is $\dfrac{-}{- + -}$ −ve

$x = 1.1$ $y$ is $\dfrac{-}{- + +}$ +ve

$x = 2.9$ $y$ is $\dfrac{+}{- + +}$ −ve

$x = 3.1$ $y$ is $\dfrac{+}{+ + +}$ +ve

9) $x^3 + Ax^2 + Bx + 15 = 0$

Root $x = 1 + 2j$

i) Other complex root $x = 1 - 2j$

ii) Complex roots occur in conjugate pairs so remaining root must be real.

Alternatively,
A cubic is a continuous function which $\to -\infty$ as $x \to -\infty$ and $\to \infty$ as $x \to \infty$ provided that coefficient of $x^3$ is $> 0$. Thus function must cross $x$-axis at some point and therefore there is a real root.

iii) There are several ways to do this!

Let real root be $\alpha$

$\Sigma\alpha = \alpha + 1 + 2j + 1 - 2j = \alpha + 2 = -A$ ①

$\Sigma\alpha\beta = \alpha(1+2j) + \alpha(1-2j) + (1+2j)(1-2j)$

$= \alpha + \cancel{2\alpha j} + \alpha - \cancel{2\alpha j} + 1 + 4$

$= 2\alpha + 5 = B$ ②

9iii cont)

$$\Sigma \alpha\beta\gamma = \alpha(1+2j)(1-2j)$$
$$= 5\alpha = -15 \quad ③$$

From ③ $\alpha = -3$

Subst in ① $-3+2 = -A$
$-1 = -A$
$\Rightarrow A = 1$

Subst in ②
$2(-3)+5 = B$
$-6+5 = B$
$-1 = B$

Solution:
Real root $x = -3$
$A = 1$, $B = -1$

10) i)
$$\underline{A} = \begin{pmatrix} 1 & -2 & k \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix}$$

$$\underline{B} = \begin{pmatrix} -5 & -2+2k & -4-k \\ 8 & -1-3k & -2+2k \\ 1 & -8 & 5 \end{pmatrix}$$

Top left element of $\underline{A}\,\underline{B}$
$= 1(-5)+(-2)8+k(1)$
$= -5-16+k$
$= k-21$
$\therefore n = 21$

ii) For inverse to exist
$k \neq 21$

$$A^{-1} = \frac{1}{k-21}\begin{pmatrix} -5 & -2+2k & -4-k \\ 8 & -1-3k & -2+2k \\ 1 & -8 & 5 \end{pmatrix}$$

iii) Let $k=1$ then

$$\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 12 \\ 3 \end{pmatrix}$$

$$\frac{1}{1-21}\begin{pmatrix} -5 & 0 & -5 \\ 8 & -4 & 0 \\ 1 & -8 & 5 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \frac{1}{1-21}\begin{pmatrix} -5 & 0 & -5 \\ 8 & -4 & 0 \\ 1 & -8 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 12 \\ 3 \end{pmatrix}$$

$$I_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-20}\begin{pmatrix} -5+0-15 \\ 8-48+0 \\ 1-96+15 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{20}\begin{pmatrix} -20 \\ -40 \\ -80 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$$

$x = 1$, $y = 2$, $z = 4$