1)

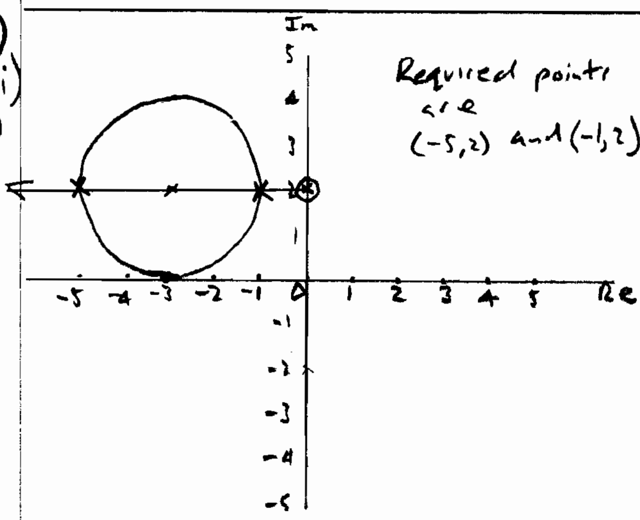
i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$

Answer $= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

ii) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

iii) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$

2) i) ii) iii)



Required points are $(-5,2)$ and $(-1,2)$

3)

$\begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \quad -x - y = x$

and $\quad 2x + 2y = y$

Invariant line

$y = -2x$

4)

$3x^3 - x^2 + 2 \equiv A(x-1)^3 + (x^3 + Bx^2 + Cx + D)$

$3x^3 - x^2 + 2 \equiv$

$A(x^3 - 3x^2 + 3x - 1) + x^3 + Bx^2 + Cx + D$

Coeff of x^3

$3 = A + 1 \quad \Rightarrow A = 2$

Coeff of x^2

$-1 = -3A + B$

$-1 = -6 + B$

$\Rightarrow B = 5$

Coeff of x

$0 = 3A + C$

$0 = 6 + C \quad \Rightarrow C = -6$

Coeff of constant

$2 = -A + D$

$2 = -2 + D$

$\Rightarrow D = 4$

$A = 2, \quad B = 5, \quad C = -6, \quad D = 4$

5)

i) $\underline{\underline{AB}} = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 2 & 5 \\ 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 & 2 \\ 14 & -14 & 7 \\ -5 & 7 & -4 \end{pmatrix}$

$= \begin{pmatrix} -1+28-20 & 0-28+28 & 2+14-16 \\ -3+28-25 & 0-28+35 & 6+14-20 \\ -4+14-10 & 0-14+14 & 8+7-8 \end{pmatrix}$

$= \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$

5ii)

$$A^{-1} = \frac{1}{7}\underline{B}$$

$$= \frac{1}{7}\begin{pmatrix} -1 & 0 & 2 \\ 14 & -14 & 7 \\ -5 & 7 & -4 \end{pmatrix}$$

or

$$\begin{pmatrix} -\frac{1}{7} & 0 & \frac{2}{7} \\ 2 & -2 & 1 \\ -\frac{5}{7} & 1 & -\frac{4}{7} \end{pmatrix}$$

6)

$$2x^3 + x^2 - 3x + 1 = 0$$

Let $w = 2x$

$$\Rightarrow \quad x = \frac{w}{2}$$

$$2\left(\frac{w}{2}\right)^3 + \left(\frac{w}{2}\right)^2 - 3\left(\frac{w}{2}\right) + 1 = 0$$

$$\frac{w^3}{4} + \frac{w^2}{4} - \frac{3w}{2} + 1 = 0$$

$$w^3 + w^2 - 6w + 4 = 0$$

7)

i) $$\frac{1}{3r-1} - \frac{1}{3r+2}$$

$$\equiv \frac{(3r+2)-(3r-1)}{(3r-1)(3r+2)}$$

$$\equiv \frac{3r+2-3r+1}{(3r-1)(3r+2)}$$

$$\equiv \frac{3}{(3r-1)(3r+2)}$$

ii) $$\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)}$$

$$= \sum_{r=1}^{n} \left(\frac{1}{3r-1} - \frac{1}{3r+2}\right)$$

=

$r=1$ $\quad + \frac{1}{2} - \cancel{\frac{1}{5}}$

$r=2$ $\quad \cancel{+\frac{1}{5}} - \cancel{\frac{1}{8}}$

$r=3$ $\quad \cancel{+\frac{1}{8}} - \cancel{\frac{1}{11}}$

$\vdots$

$r=n-1$ $\quad \cancel{+\frac{1}{3(n-1)-1}} - \cancel{\frac{1}{3(n-1)+2}}$

$r=n$ $\quad \cancel{+\frac{1}{3n-1}} - \frac{1}{3n+2}$

$$= \frac{1}{2} - \frac{1}{3n+2}$$

8)

$$y = \frac{2x^2}{(x-3)(x+2)}$$

i) Asymptotes $\quad x = 3$, $\quad x = -2$, $\quad y = 2$

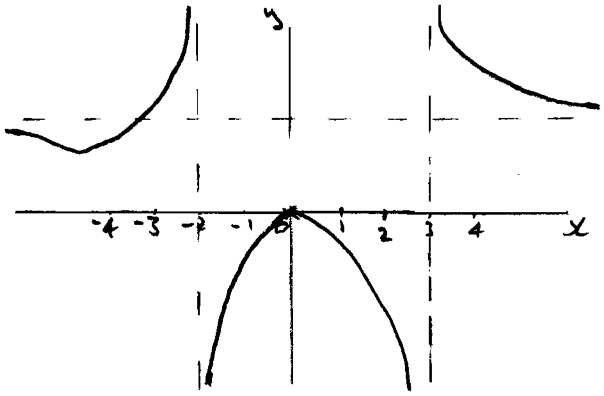
ii) when $x = 100$ $\quad y = \frac{2 \times 100^2}{97 \times 102} = 2.02$

As $x \to \infty$, $y \to 2^+$

when $x = -100$ $\quad y = \frac{2 \times (-100)^2}{-103 \times -98} = 1.98$

As $x \to -\infty$, $y \to 2^-$

8 iii)



When $x = 0$, $y = 0$

when $x = -2.1$ y is $\frac{-^2}{- -} = +ve$

when $x = -1.9$ y is $\frac{-^2}{- +} = -ve$

when $x = 2.9$ y is $\frac{+^2}{- +} = -ve$

when $x = 3.1$ y is $\frac{+^2}{+ +} = +ve$

8 iv) From graph

$-2 < x < 0$

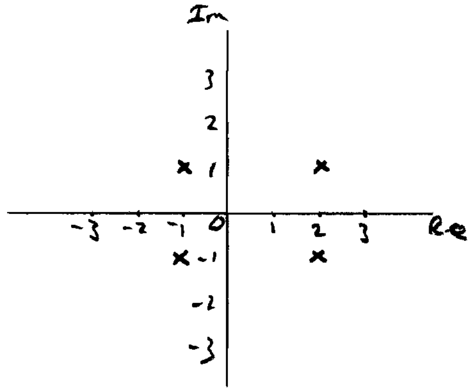
or $0 < x < 3$

9) $\alpha = 2 - j$ $\quad \beta = -1 + j$

i) Other roots

$\gamma = 2 + j$ $\quad \delta = -1 - j$

ii)



iii)
$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$$
$$(x-(2-j))(x-(2+j))(x-(-1+j))(x-(-1-j)) = 0$$
$$((x-2)+j)((x-2)-j)((x+1)-j)((x+1)+j) = 0$$
$$((x-2)^2+1^2)((x+1)^2+1^2) = 0$$
$$(x^2-4x+5)(x^2+2x+2) = 0$$
$$x^4 - 4x^3 + 5x^2 + 2x^3 - 8x^2 + 10x + 2x^2 - 8x + 10 = 0$$
$$x^4 - 2x^3 - x^2 + 2x + 10 = 0$$

$A = -2$, $B = -1$, $C = 2$, $D = 10$

iii) Alternative method

$$\Sigma\alpha = 2+j+2-j+(-1+j)+(-1-j)$$
$$= 2+j+2-j-1+j-1-j$$
$$= 2$$
$$\Rightarrow A = -2$$

$$\alpha\beta\gamma\delta = (2+j)(2-j)(-1+j)(-1-j)$$
$$= (2^2+1^2)(1^2+1^2)$$
$$= 5 \times 2 = 10$$
$$\Rightarrow D = 10$$

$$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

9 iii) cont)

$= (2+j)(2-j) + (2+j)(-1+j)$

$+ (2+j)(-1-j) + (2-j)(-1+j)$

$+ (2-j)(-1-j) + (-1+j)(-1-j)$

$= 2^2+1^2 - 2 \cancel{+j} - 1 - 2 \cancel{-3j} + 1$

$-2 \cancel{+3j} + 1 - 2 \cancel{-j} - 1 + 1^2 + 1^2$

$= 4 + 1 - 2 - 1 - 2 + 1 - 2$

$+ 1 - 2 - 1 + 1 + 1$

$= -1$

$\Rightarrow \beta = -1$

Consider $\sum \alpha\beta\gamma$

$= (2+j)(2-j)(-1+j)$

$+ (2+j)(2-j)(-1-j)$

$+ (2-j)(-1+j)(-1-j)$

$+ (2+j)(-1+j)(-1-j)$

$= 5(-1+j)$

$+ 5(-1-j)$

$+ (2-j)\times 2$

$+ (2+j)\times 2$

$= -5 \cancel{+5j} - 5 \cancel{-5j}$

$+4 \cancel{-2j} + 4 \cancel{+2j}$

$= -2$

$\Rightarrow c = +2$

$\therefore A = -2,\ B = -1,\ C = +2,\ D = +10$

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10) i)

$\sum_{r=1}^{n} r^2(r+1)$

$= \sum_{r=1}^{n} (r^3 + r^2)$

$= \frac{1}{4}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1)$

$= \frac{1}{12}n(n+1)\left[3n(n+1) + 2(2n+1)\right]$

$= \frac{1}{12}n(n+1)\left[3n^2 + 3n + 4n + 2\right]$

$= \frac{1}{12}n(n+1)(3n^2 + 7n + 2)$

$= \frac{1}{12}n(n+1)(n+2)(3n+1)$

---

10 ii)

Prove true for $n = 1$

$\sum_{r=1}^{1} r^2(r+1) = 1^2(1+1) = 2$

$\frac{1}{12}(1)(1+1)(1+2)(3+1)$

$= \frac{1}{12}(1)(2)(3)(4) = \frac{24}{12} = 2$ ✓

$\therefore$ formula true for $n=1$

Assume true for $n=k$

then $\sum_{r=1}^{k} r^2(r+1) =$

$\frac{1}{12}k(k+1)(k+2)(3k+1)$

(Demonstrate formula would also then be true for $n = k+1$)

10ii cont) $\Rightarrow \displaystyle\sum_{r=1}^{k+1} r^2(r+1) = \tfrac{1}{12}k(k+1)(k+2)(3k+1) + (k+1)^2(k+1+1)$

$$= \frac{1}{12}(k+1)(k+2)\left[k(3k+1) + 12(k+1)\right]$$

$$= \frac{1}{12}(k+1)(k+2)\left(3k^2 + k + 12k + 12\right)$$

$$= \frac{1}{12}(k+1)(k+2)\left(3k^2 + 13k + 12\right)$$

$$= \frac{1}{12}(k+1)(k+2)(k+3)(3k+4)$$

$$= \frac{1}{12}(k+1)\big((k+1)+1\big)\big((k+2)+1\big)\big(3(k+1)+1\big)$$

which is the same formula with k replaced by k+1

$\therefore$ if formula is true for $n=k$, it is also true for $n=k+1$.

Since formula is true for $n=1$, by mathematical induction it is true for all positive integers n