MEI FP1 JUNE 2009

1)

i)

$$\underline{M} = \begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}$$

$$\det \underline{M} = (4\times 2) - (3\times -1)$$
$$= 8 + 3 = 11$$

$$\underline{M}^{-1} = \frac{1}{11}\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{11} & \frac{1}{11} \\ -\frac{3}{11} & \frac{4}{11} \end{pmatrix}$$

ii)

$$4x - y = 49$$
$$3x + 2y = 100$$

$$\begin{pmatrix} 4 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 49 \\ 100 \end{pmatrix}$$

$$\underline{M}^{-1}\underline{M}\begin{pmatrix} x \\ y \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} 49 \\ 100 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 2 & 1 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 49 \\ 100 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 198 \\ 253 \end{pmatrix} = \begin{pmatrix} 18 \\ 23 \end{pmatrix}$$

$$x = 18, \quad y = 23$$

2) $z^3 + z^2 - 7z - 15 = 0$

Sub $z = 3$

$$3^3 + 3^2 - 7(3) - 15$$
$$= 27 + 9 - 21 - 15 = 0$$

$\therefore$ $z = 3$ is a root.

$$\begin{array}{r|l} & z^2 + 4z + 5 \\ \hline z-3 & z^3 + z^2 - 7z - 15 \\ & \underline{z^3 - 3z^2} \\ & \quad 4z^2 - 7z \\ & \quad \underline{4z^2 - 12z} \\ & \qquad 5z - 15 \\ & \qquad \underline{5z - 15} \end{array}$$

$$(z-3)(z^2 + 4z + 5) = 0$$

$$z = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$z = \frac{-4 \pm \sqrt{-4}}{2}$$

$$z = \frac{-4 \pm 2j}{2}$$

$$z = -2 \pm j$$

Roots are $z = 3$

$z = -2 + j$

$z = -2 - j$

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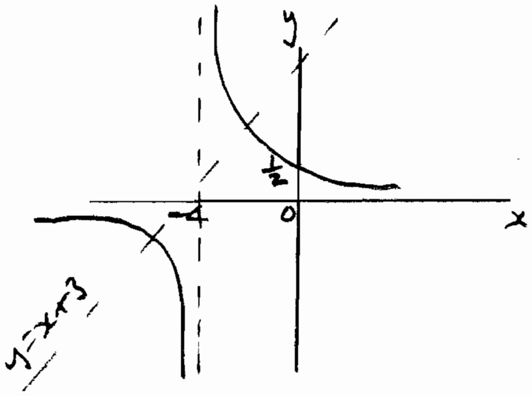
3)

i)



ii) $$\frac{2}{x+4} \le x+3$$

Solve $\frac{2}{x+4} = x+3$

$$2 = (x+3)(x+4)$$

$$2 = x^2 + 7x + 12$$

$$0 = x^2 + 7x + 10$$

$$0 = (x+2)(x+5)$$

$$\Rightarrow x = -2 \text{ or } x = -5$$

Now use intersection of graphs to solve

$$\frac{2}{x+4} \le x+3$$

$$-5 \le x < -4$$

or $$x \geq -2$$

4) $2x^3 + x^2 + px + q = 0$

Roots $2w, -6w, 3w$

$\Sigma\alpha = 2w + 3w - 6w = -w$

$\therefore -w = -\frac{1}{2}$

$\Rightarrow w = \frac{1}{2}$

Roots are $1, \frac{3}{2}, -3$

$\Sigma\alpha\beta = (1 \times \frac{3}{2}) + (1 \times -3) + (\frac{3}{2} \times -3)$

$= \frac{3}{2} - 3 - \frac{9}{2} = -6$

$\therefore -6 = \frac{p}{2}$

$\Rightarrow p = -12$

$\alpha\beta\gamma = 1 \times \frac{3}{2} \times (-3) = -\frac{9}{2}$

$\Rightarrow -\frac{q}{2} = -\frac{9}{2}$

$\Rightarrow q = 9$

$\therefore$ roots are $1, \frac{3}{2}, -3$

$p = -12$

$q = 9$

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5) i)

$$\frac{1}{5r-2} - \frac{1}{5r+3}$$

$$\equiv \frac{5r+3-(5r-2)}{(5r-2)(5r+3)}$$

$$\equiv \frac{5r+3-5r+2}{(5r-2)(5r+3)}$$

$$\equiv \frac{5}{(5r-2)(5r+3)}$$

ii)

$$\sum_{r=1}^{n} \frac{1}{(5r-2)(5r+3)} = \frac{1}{5}\left[\frac{1}{5r-2} - \frac{1}{5r+3}\right]$$

$$= \frac{1}{5}\left[\begin{array}{cccc} r & \frac{1}{5r-2} & - & \frac{1}{5r+3} \\ 1 & \frac{1}{3} & - & \cancel{\frac{1}{8}} \\ 2 & \cancel{\frac{1}{8}} & - & \cancel{\frac{1}{13}} \\ 3 & \cancel{\frac{1}{13}} & - & \cancel{\frac{1}{18}} \\ \vdots & & & \\ n-1 & \cancel{\frac{1}{5n-7}} & - & \cancel{\frac{1}{5n-2}} \\ n & \cancel{\frac{1}{5n-2}} & - & \frac{1}{5n+3} \end{array}\right]$$

$$= \frac{1}{5}\left[\frac{1}{3} - \frac{1}{5n+3}\right]$$

$$= \frac{1}{5}\left[\frac{5n+3-3}{3(5n+3)}\right]$$

$$= \frac{1}{5}\left[\frac{5n}{3(5n+3)}\right]$$

$$= \frac{n}{3(5n+3)}$$

6) Prove

$$3+10+17+\dots+(7n-4) = \frac{1}{2}n(7n-1)$$

When $n=1$

$$3 = \frac{1}{2}(1)(7-1) = 3 \quad ✓$$

$\therefore$ true when $n=1$

Assume true for $n=k$

$$3+10+\dots+(7k-4) = \frac{1}{2}k(7k-1)$$

Then consider

$$3+10+\dots+(7k-4)+(7(k+1)-4)$$

$$= \frac{1}{2}k(7k-1) + 7(k+1)-4$$

$$= \frac{1}{2}\left[7k^2 - k + 14k + 14 - 8\right]$$

$$= \frac{1}{2}\left[7k^2 + 13k + 6\right]$$

$$= \frac{1}{2}\left[(7k+6)(k+1)\right]$$

$$= \frac{1}{2}(k+1)(7(k+1)-1)$$

This is same formula with k replaced by $k+1$. Therefore if true for $n=k$ it is also true for $n=k+1$

Since it is true for $n=1$, by mathematical induction it is true for all positive integers n.

7) $$y = \frac{(x+2)(3x-5)}{(2x+1)(x-1)}$$

i) When $x=0$, $y = \frac{2(-5)}{1(-1)} = 10$

Crosses y-axis at $(0, 10)$

When $y=0$, $x=-2$ or $x=\frac{5}{3}$

Crosses x-axis at

$(-2, 0)$ and $(5/3, 0)$

ii) Asymptotes:

$x = -\frac{1}{2}$

$x = 1$

$y = \frac{3}{2}$

iii) When $x = 100$

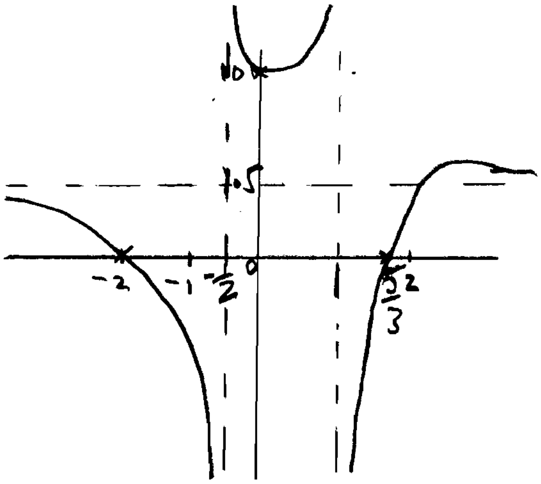
$y = \frac{102 \times 295}{201 \times 99} = 1.51\ldots$

$y \to 1.5^{+}$

When $x = -100$

$y = \frac{-98 \times -305}{-199 \times -101} = 1.48\ldots$

$y \to 1.5^{-}$



8) i) $|z-(4+2j)| = 2$

ii) $\arg(z-(4+2j)) = 0$

iii) $P = 4+2j + 2\left(\cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4}\right)$

$P = 4+2j - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}j$

$P = 4+2j - \sqrt{2} + \sqrt{2}j$

$P = (4-\sqrt{2}) + (2+\sqrt{2})j$

iv) $|z-(4+2j)| < 2$

and $0 < \arg(z-(4+2j)) < \frac{3\pi}{4}$

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9) i) $\underline{M} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ $\underline{N} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

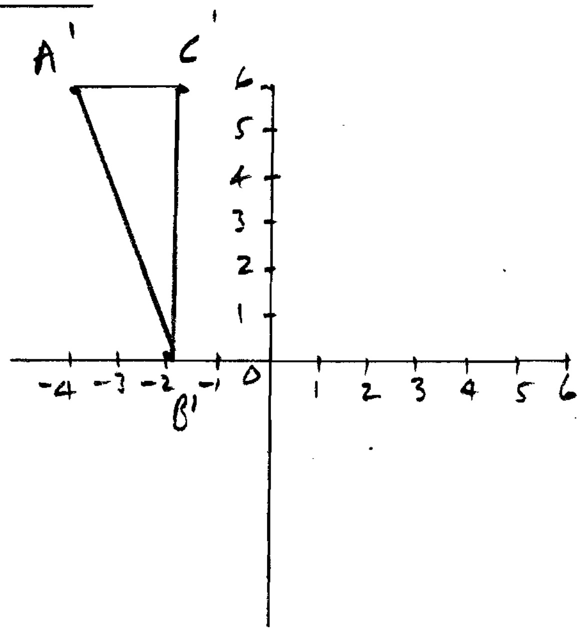
Matrix multiplication is associative

QMN

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$

$= \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$



$$\begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 1 & 2 \\ 2 & 0 & 2 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} A' & B' & C' \end{array} \\ \begin{pmatrix} -2 & -2 & -4 \\ 6 & 0 & 6 \end{pmatrix} \end{array}$$

ii) $\underline{M}$ represents a two-way stretch, by a scale factor of 3 parallel to x-direction and scale factor 2 parallel to y-direction.

$\underline{N}$ represents a reflection in line $y = x$

$\underline{Q}$ represents a rotation by 90° anti-clockwise about $(0,0)$