1 a) $r = a(\sqrt{2} + 2\cos\theta)$

i)

$\theta = 0 \quad \pm\frac{\pi}{6} \quad \pm\frac{\pi}{4} \quad \pm\frac{\pi}{3} \quad \pm\frac{\pi}{2} \quad \pm\frac{2\pi}{3} \quad \pm\frac{3\pi}{4}$

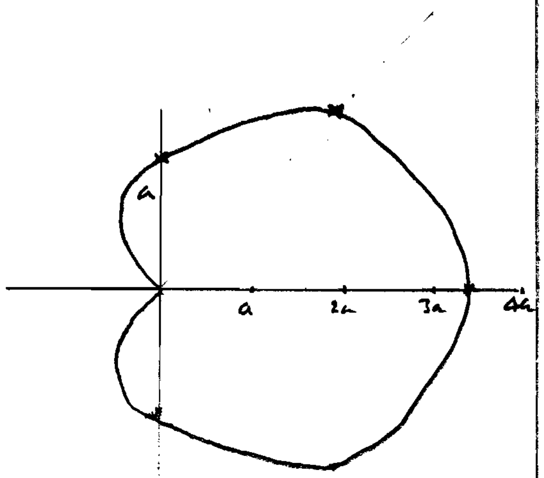
$\cos\theta = 1 \quad \frac{\sqrt{3}}{2} \quad \frac{1}{\sqrt{2}} \quad \frac{1}{2} \quad 0 \quad \frac{1}{2} \quad \frac{1}{\sqrt{2}}$

$\theta = 0, \ r \approx 3.4a$

$\theta = \pm\frac{\pi}{4} \quad r \approx 2.8a$

$\theta = \pm\frac{\pi}{2} \quad r \approx 1.4a$

$\theta = \pm\frac{3\pi}{4} \quad r = 0$



ii)

$$\text{Area} = \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} r^2 \, d\theta$$

$$= \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} a^2 (\sqrt{2} + 2\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2} a^2 \int (2 + 4\sqrt{2}\cos\theta + 4\cos^2\theta) \, d\theta$$

$$= a^2 \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} (1 + 2\sqrt{2}\cos\theta + 2\cos^2\theta) \, d\theta$$

(Note $2\cos^2\theta = 1 + \cos 2\theta$)

$$= a^2 \int_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}} (2 + 2\sqrt{2}\cos\theta + \cos 2\theta) \, d\theta$$

$$= a^2 \left[2\theta + 2\sqrt{2}\sin\theta + \frac{1}{2}\sin 2\theta \right]_{-\frac{3\pi}{4}}^{\frac{3\pi}{4}}$$

$$= a^2 \left[\left(\frac{3\pi}{2} + 2 - \frac{1}{2} \right) - \left(-\frac{3\pi}{2} - 2 + \frac{1}{2} \right) \right]$$

$$= a^2 \left[3\pi + 4 - 1 \right]$$

$$= a^2 (3\pi + 3)$$

$$= 3(\pi + 1)a^2$$

1b) i)

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(0)$$

$$f(x) = \tan\left(\frac{\pi}{4}+x\right)$$

$$f(0) = \tan\frac{\pi}{4} = 1$$

$$f'(x) = \sec^2\left(\frac{\pi}{4}+x\right) = \frac{1}{\cos^2\left(\frac{\pi}{4}+x\right)}$$

$$f'(0) = \frac{1}{\cos^2\frac{\pi}{4}}$$

$$f'(0) = \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2} = 2$$

Now $f'(x) = \left(\cos\left(\frac{\pi}{4}+x\right)\right)^{-2}$

$$\Rightarrow f''(x) = -2\left(\cos\left(\frac{\pi}{4}+x\right)\right)^{-3}(-\sin x)$$

$$= \frac{+2\sin x}{\cos^3\left(x+\frac{\pi}{4}\right)}$$

$$f''(0) = \frac{2\times\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^3} = 4$$

$$\therefore f(x) \approx 1 + 2x + 4\frac{x^2}{2}$$

$$f(x) \approx 1 + 2x + 2x^2$$

1b ii)

$$\int_{-h}^{h} x^2\tan\left(\frac{\pi}{4}+x\right)dx$$

$$\approx \int_{-h}^{h} x^2(1+2x+2x^2)\,dx$$

$$= \int_{-h}^{h}(x^2+2x^3+2x^4)\,dx$$

$$= \left[\frac{x^3}{3}+\frac{x^4}{2}+\frac{2x^5}{5}\right]_{-h}^{h}$$

$$= \left(\frac{h^3}{3}+\frac{h^4}{2}+\frac{2h^5}{5}\right)-\left(-\frac{h^3}{3}+\frac{h^4}{2}-\frac{2h^5}{5}\right)$$

$$= \frac{2h^2}{3}+\frac{4h^5}{5}$$

2)

ai) $z = \cos\theta + j\sin\theta$

$z^n = (\cos\theta + j\sin\theta)^n$

$= \cos n\theta + j\sin n\theta$

$z^{-n} = \cos n\theta - j\sin n\theta$

$\therefore \quad z^n + \frac{1}{z^n} = 2\cos n\theta$

$z^n - \frac{1}{z^n} = 2j\sin n\theta$

aii) $\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2$

$= (2j\sin\theta)^4 (2\cos\theta)^2$

$= 64\sin^4\theta\cos^2\theta$

But we also have:

$\left(z - \frac{1}{z}\right)^4 \left(z + \frac{1}{z}\right)^2$

$= \left[z^4 - 4z^3\frac{1}{z} + 6z^2\frac{1}{z^2} - 4z\frac{1}{z^3} + \frac{1}{z^4}\right] \times \left[z^2 + 2z\frac{1}{z} + \frac{1}{z^2}\right]$

$= \left[z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right] \times \left[z^2 + 2 + \frac{1}{z^2}\right]$

$= z^6 - 4z^4 + 6z^2 - 4 + \frac{1}{z^2}$
$\quad + 2z^4 - 8z^2 + 12 - \frac{8}{z^2} + \frac{2}{z^4}$
$\quad + z^2 - 4 + \frac{6}{z^2} - \frac{4}{z^4} + \frac{1}{z^6}$

$= \left(z^6 + \frac{1}{z^6}\right) - 2\left(z^4 + \frac{1}{z^4}\right) - \left(z^2 + \frac{1}{z^2}\right) + 4$

$= 2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4$

$\therefore \quad \sin^4\theta\cos^2\theta$

$= \frac{2\cos 6\theta - 4\cos 4\theta - 2\cos 2\theta + 4}{64}$

$= \frac{1}{32}\cos 6\theta - \frac{1}{16}\cos 4\theta - \frac{1}{32}\cos 2\theta + \frac{1}{16}$

$\therefore A = \frac{1}{32},\ B = -\frac{1}{16},\ C = -\frac{1}{32},\ D = \frac{1}{16}$

2bi) $|4 + 4j| = \sqrt{4^2 + 4^2}$

$= \sqrt{32} = 4\sqrt{2}$

$\arg(4 + 4j) = \tan^{-1}\frac{4}{4}$

$= \frac{\pi}{4}$

2bii)

$4 + 4j = \sqrt{32}\, e^{j\frac{\pi}{4}}$

$= \sqrt{2}^5\, e^{j\frac{\pi}{4}}$

2bii) cont

5^{th} roots of $4+4j$ are given by

$$\sqrt{2}\, e^{j\left(\frac{\frac{\pi}{4}+2\pi n}{5}\right)}$$

for $n = 0, 1, 2, 3, 4$

$$= \sqrt{2}\, e^{j\left(\frac{\pi + 8\pi n}{20}\right)}$$

$$= \sqrt{2}\, e^{j\frac{\pi}{20}},\ \sqrt{2}\, e^{j\frac{9\pi}{20}},\ \sqrt{2}\, e^{j\frac{17\pi}{20}}$$

$$\sqrt{2}\, e^{j\frac{25\pi}{20}},\ \sqrt{2}\, e^{j\frac{33\pi}{20}}$$

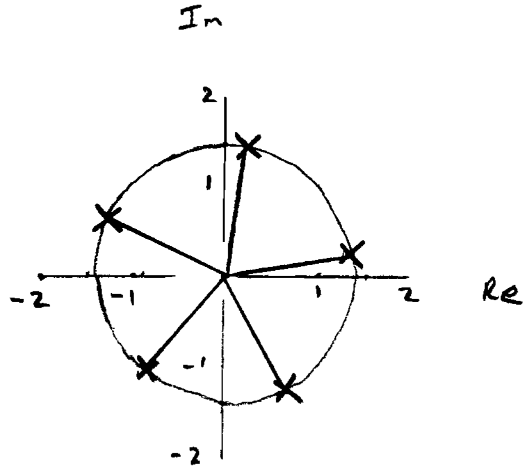
$$\downarrow \qquad \downarrow$$

$$\sqrt{2}\, e^{-j\frac{15\pi}{20}} \qquad \sqrt{2}\, e^{-j\frac{7\pi}{20}}$$

So for roots in form $re^{j\theta}$

$r = \sqrt{2}$

$\theta = -\frac{15\pi}{20}, -\frac{7\pi}{20}, \frac{\pi}{20}, \frac{9\pi}{20}, \frac{17\pi}{20}$



$\frac{\pi}{20} = 9°$

so measure angles from real axis

$9°, 81°, 153°, 224°, 295°$

All roots are on circle centre the origin, radius $\sqrt{2}$

2biii)

Find p and q such that $(p+jq)^5 = 4+4j$

The accurate argand diagram suggests the root where p and q are integers is $(-1-1j)$

so $p = -1$, $q = -1$

Check

$$(p+jq)^5 = p^5 + 5p^4qj + 10p^3q^2j^2 + 10p^2q^3j^3 + 5pq^4j^4 + q^5j^5$$

$$= p^5 + 5p^4qj - 10p^3q^2 - 10p^2q^3j + 5pq^4 + q^5j$$

Subst $p = -1$, $q = -1$

$$= -1 - 5j + 10 + 10j - 5 - j$$

$$= 4 + 4j \ ✓$$

$\therefore p = -1$, $q = -1$

3i) Let $\underline{M} = \begin{pmatrix} 4 & 1 & k \\ 3 & 2 & 5 \\ 8 & 5 & 13 \end{pmatrix}$

$\det \underline{M} = 4(26-25) - 1(39-40) + k(15-16)$

$\det \underline{M} = 4 + 1 - k = 5 - k$

Calculate unsigned minors

$$\begin{pmatrix} 1 & -1 & -1 \\ 13-5k & 52-8k & 12 \\ 5-2k & 20-3k & 5 \end{pmatrix}$$

Adjust signs according to $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$ to give cofactors

$$\begin{pmatrix} 1 & 1 & -1 \\ 5k-13 & 52-8k & -12 \\ 5-2k & 3k-20 & 5 \end{pmatrix}$$

Transpose to give

$$\operatorname{adj} \underline{M} = \begin{pmatrix} 1 & 5k-13 & 5-2k \\ 1 & 52-8k & 3k-20 \\ -1 & -12 & 5 \end{pmatrix}$$

$$\underline{M}^{-1} = \frac{1}{5-k}\begin{pmatrix} 1 & 5k-13 & 5-2k \\ 1 & 52-8k & 3k-20 \\ -1 & -12 & 5 \end{pmatrix}$$

3ii) If $k = 7$

$$\underline{M}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ m \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{M}^{-1}\begin{pmatrix} 12 \\ m \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} 1 & 22 & -9 \\ 1 & -4 & 1 \\ -1 & -12 & 5 \end{pmatrix}\begin{pmatrix} 12 \\ m \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 12 + 22m \\ 12 - 4m \\ -12 - 12m \end{pmatrix} = \begin{pmatrix} -6 - 11m \\ -6 + 2m \\ 6 + 6m \end{pmatrix}$$

$\therefore$ $x = -6 - 11m$
$y = -6 + 2m$
$z = 6 + 6m$

3iii) No unique solution since $\det \underline{M} = 0$

$4x + y + 5z = 12$ ①
$3x + 2y + 5z = p$ ②
$8x + 5y + 13z = 0$ ③

② − 2① $\quad -5x - 5z = p - 24$ ④

③ − 5① $\quad -12x - 12z = -60$ ⑤

For consistency

$$-60 = \frac{12}{5}(p - 24)$$

$$-60 \times \frac{5}{12} = p - 24$$

$$-25 = p - 24$$

$$-1 = p$$

$$\therefore p = -1$$

3 iii) cont) From 5

$$-x - z = -5$$

$$5 - z = x$$

Subst for x in 3

$$8(5-z) + 5y + 13z = 0$$

$$40 - 8z + 5y + 13z = 0$$

$$5y + 5z = -40$$

$$5y = -40 - 5z$$

$$y = -8 - z$$

In solution let $z = \lambda$

then

$$x = 5 - \lambda$$

$$y = -8 - \lambda$$

$$z = \lambda$$

4 i)

$$\sinh x = \tfrac{1}{2}\left(e^{x} - e^{-x}\right)$$
$$\cosh x = \tfrac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$1 + 2\sinh^2 x$$
$$= 1 + 2\left(\tfrac{1}{2}\left(e^{x} - e^{-x}\right)\right)^2$$
$$= 1 + \tfrac{1}{2}\left(e^{x} - e^{-x}\right)\left(e^{x} - e^{-x}\right)$$
$$= 1 + \tfrac{1}{2}\left[e^{2x} - 2 + e^{-2x}\right]$$
$$= 1 - 1 + \tfrac{1}{2}\left(e^{2x} + e^{-2x}\right)$$
$$= \cosh(2x)$$

---

4 ii)

$$2\cosh 2x + \sinh x = 5$$
$$2\left(1 + 2\sinh^2 x\right) + \sinh x = 5$$
$$2 + 4\sinh^2 x + \sinh x = 5$$
$$4\sinh^2 x + \sinh x - 3 = 0$$
$$(4\sinh x - 3)(\sinh x + 1) = 0$$
$$\Rightarrow \sinh x = \tfrac{4}{3} \text{ or } \sinh x = -1$$
$$\Rightarrow x = \operatorname{arsinh}\left(\tfrac{4}{3}\right) \text{ or } x = \operatorname{arsinh}(-1)$$
$$\Rightarrow x = \ln\left(\tfrac{3}{4} + \sqrt{\tfrac{9}{16} + 1}\right)$$
$$= \ln\left(\tfrac{3}{4} + \sqrt{\tfrac{25}{16}}\right)$$
$$= \ln\left(\tfrac{3}{4} + \tfrac{5}{4}\right)$$
$$= \ln 2$$

$$\text{or } x = \ln\left(-1 + \sqrt{1+1}\right)$$
$$x = \ln\left(\sqrt{2} - 1\right)$$

---

4 iii)

$$\int_0^{\ln 3} \sinh^2 x \, dx$$
$$= \int_0^{\ln 3} \frac{\cosh 2x - 1}{2} \, dx$$
$$= \frac{1}{2}\int_0^{\ln 3} (\cosh 2x - 1)\, dx$$
$$= \frac{1}{2}\left[\frac{1}{2}\sinh 2x - x\right]_0^{\ln 3}$$
$$= \frac{1}{2}\left[\left(\frac{1}{2}\sinh(2\ln 3) - \ln 3\right) - \left(\frac{1}{2}\sinh 0 - 0\right)\right]$$
$$= \frac{1}{2}\left[\frac{1}{2}\sinh(\ln 9) - \ln 3\right]$$
$$= \frac{1}{2}\left[\frac{1}{2} \times \frac{1}{2}\left(e^{\ln 9} - e^{\ln 9}\right) - \ln 3\right]$$
$$= \frac{1}{8}\left[9 - \frac{1}{9}\right] - \frac{1}{2}\ln 3$$
$$= \frac{1}{8} \times \frac{80}{9} - \frac{1}{2}\ln 3$$
$$= \frac{10}{9} - \frac{1}{2}\ln 3$$

4iii) alternative method

$$\int_0^{\ln 3} \sinh^2 x \, dx$$

$$= \int_0^{\ln 3} \left(\tfrac{1}{2}(e^x - e^{-x})\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\ln 3} (e^x - e^{-x})^2 dx$$

$$= \frac{1}{4}\int_0^{\ln 3} (e^{2x} - 2 + e^{-2x}) dx$$

$$= \frac{1}{4}\left[\frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x}\right]_0^{\ln 3}$$

$$= \frac{1}{8}\left[e^{2x} - 4x - e^{-2x}\right]_0^{\ln 3}$$

$$= \frac{1}{8}\left[\left(e^{2\ln 3} - 4\ln 3 - e^{-2\ln 3}\right) - \left(e^0 - 0 - e^0\right)\right]$$

$$= \frac{1}{8}\left[9 - 4\ln 3 - \frac{1}{9}\right]$$

$$= \frac{1}{8}\left[\frac{80}{9} - 4\ln 3\right]$$

$$= \frac{10}{9} - \frac{1}{2}\ln 3$$

4iv

$$\int_3^5 \sqrt{x^2 - 9} \, dx$$

Let $3\cosh u = x$

$$\Rightarrow 3\sinh u = \frac{dx}{du}$$

$$3\sinh u \, du = dx$$

$\cosh u = \frac{x}{3}$

Limits $x = 5 \Rightarrow u = \text{arcosh}\left(\frac{5}{3}\right)$

$x = 3 \Rightarrow u = \text{arcosh}(1) = 0$

$$\text{arcosh}(5/3) = \ln\left(\frac{5}{3} + \sqrt{\frac{25}{9} - 1}\right)$$

$$= \ln\left(\frac{5}{3} + \frac{4}{3}\right) = \ln 3$$

Integral becomes

$$\int_0^{\ln 3} \sqrt{9\cosh^2 u - 9} \times 3\sinh u \, du$$

$$= \int_0^{\ln 3} 3\sqrt{\cosh^2 u - 1} \times 3\sinh u \, du$$

$$= 9\int_0^{\ln 3} \sqrt{\sinh^2 u} \times \sinh u \, du$$

$$= 9\int_0^{\ln 3} \sinh^2 u \, du$$

$$= 9\int_0^{\ln 3} \left(\tfrac{1}{2}(e^u - e^{-u})\right)^2 du$$

$$= \frac{9}{4}\int_0^{\ln 3} (e^{2u} - 2 + e^{-2u}) du$$

$$= \frac{9}{4}\left[\frac{1}{2}e^{2u} - 2u - \frac{1}{2}e^{-2u}\right]_0^{\ln 3}$$

$$= \frac{9}{8}\left[e^{2u} - 4u - e^{-2u}\right]_0^{\ln 3}$$

$$= \frac{9}{8}\left[e^{2\ln 3} - 4\ln 3 - e^{-2\ln 3}\right]_0^{\ln 3}$$

$$= \frac{9}{8}\left[9 - 4\ln 3 - \frac{1}{9}\right]$$

$$= \frac{9}{8}\left[\frac{80}{9} - 4\ln 3\right]$$

$$= 10 - \frac{9}{2}\ln 3$$