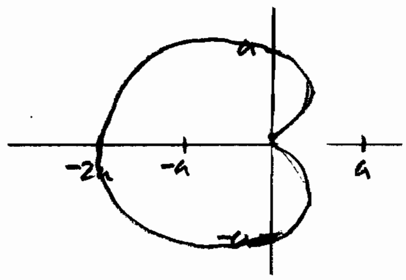
1) a) $r = a(1-\cos\theta)$

i)

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	0	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

for $-\pi \le \theta \le 0$
$\cos(-\theta) = \cos\theta$
so r the same as above



ii) $$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\, d\theta$$

$$= \frac{1}{2}\int a^2(1-\cos\theta)^2\, d\theta$$

$$= \frac{1}{2}a^2\int_0^{\frac{\pi}{2}}(1-2\cos\theta+\cos^2\theta)\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\left(1-2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}}\left(\frac{3}{2}-2\cos\theta+\frac{1}{2}\cos 2\theta\right)d\theta$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2}-2\sin\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2}\left[\left(\frac{3\pi}{4}-2+0\right)-(0-0+0)\right]$$

$$= \frac{a^2}{2}\left(\frac{3\pi}{4}-2\right)$$

b) $$\int_0^1 \frac{1}{(4-x^2)^{3/2}}\, dx$$

Let $x = 2\sin u$

$$\Rightarrow \frac{dx}{du} = 2\cos u$$

$$dx = 2\cos u\, du$$

when $x = 1$ $u = \frac{\pi}{6}$

when $x = 0$ $u = 0$

Integral becomes

$$\int_0^{\frac{\pi}{6}} \frac{1}{(4-4\sin^2 u)^{3/2}} \times 2\cos u\, du$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{8(1-\sin^2 u)^{3/2}} \times 2\cos u\, du$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \frac{\cos u}{\cos^3 u}\, du$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \frac{1}{\cos^2 u}\, du$$

1b) (cont)

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \sec^2 u \, du$$

$$= \frac{1}{4}\Big[\tan u\Big]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{3}} - 0\right]$$

$$= \frac{1}{4\sqrt{3}} \quad \text{as required}$$

1c) $f(x) = \cos^{-1}(2x)$

i) Let $y = \cos^{-1}(2x)$

$$\Rightarrow \cos y = 2x$$

$$\Rightarrow -\sin y \frac{dy}{dx} = 2$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{\sin y}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{\sqrt{1-\cos^2 y}}$$

$$\Rightarrow f'(x) = -\frac{2}{\sqrt{1-4x^2}}$$

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ii) $f'(x) = -2(1-4x^2)^{-\frac{1}{2}}$

$$f'(x) \approx -2\left[1 + -\tfrac{1}{2}(-4x^2) + \frac{-\frac{1}{2}\cdot -\frac{3}{2}}{1\cdot 2}(-4x^2)^2\right]$$

$$f'(x) \approx -2\left[1 + 2x^2 + 6x^4\right]$$

$$f'(x) \approx -2 - 4x^2 - 12x^4$$

$$\Rightarrow f(x) \approx -2x - \frac{4x^3}{3} - \frac{12x^5}{5} + c$$

When $x = 0$

$$f(x) = \cos^{-1}(0) = \frac{\pi}{2}$$

$$\therefore f(x) \approx \frac{\pi}{2} - 2x - \frac{4x^3}{3} - \frac{12x^5}{5}$$

2 a) De Moivre's theorem

$$5\cos\theta + j\sin 5\theta = (\cos\theta + j\sin\theta)^5 = (c+js)^5$$

where $s = \sin\theta$, $c = \cos\theta$

$$5\cos\theta + j\sin 5\theta = c^5 + 5c^4 js - 10c^3s^2 - 10c^2 js^3 + 5cs^4 + js^5$$

Equating real and imaginary parts

$$\sin 5\theta = 5c^4 s - 10c^2 s^3 + s^5$$

$$= 5(1-s^2)^2 s - 10(1-s^2)s^3 + s^5$$

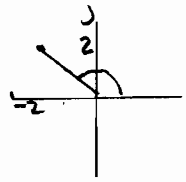
$$= 5(1-2s^2+s^4)s - 10s^3 + 10s^5 + s^5$$

$$= 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5$$

$$= 5s - 20s^3 + 16s^5$$

$$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$

2b)



$$|-2+2j| = \sqrt{2^2+2^2} = 2\sqrt{2}$$

$$\arg(-2+2j) = \frac{3\pi}{4}$$

Modulus of cubed root $= \sqrt[3]{2\sqrt{2}} = \sqrt{2}$

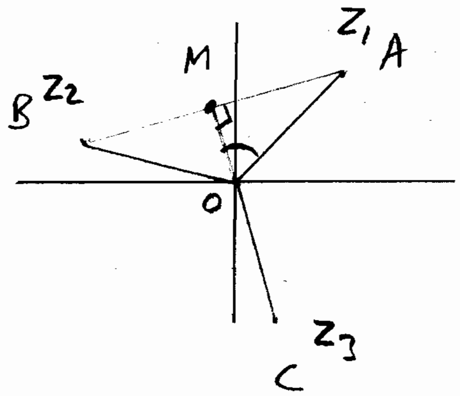
Argument of cubed root $= \left(\frac{3\pi}{4}\right) \div 3 + \frac{2n\pi}{3}$ for $n = 0, 1, 2$

$$= \frac{\pi}{4}, \frac{11\pi}{12}, \frac{19\pi}{12}$$

for $-\pi < \theta \leq \pi$ $= \frac{\pi}{4}, \frac{11\pi}{12}, -\frac{5\pi}{4}$

Cubed roots $\sqrt{2}e^{j\frac{\pi}{4}}, \sqrt{2}e^{j\frac{11\pi}{12}}, \sqrt{2}e^{-j\frac{5\pi}{12}}$

2bii)



2biii)

$$\arg w = \frac{\arg z_1 + \arg z_2}{2}$$

$$= \frac{\frac{3\pi}{12} + \frac{11\pi}{12}}{2} = \frac{7\pi}{12}$$

In $\triangle OMA$

$$\cos(\angle AOM) = \frac{|w|}{|z_1|}$$

$$\cos\left(\frac{7\pi}{12} - \frac{3\pi}{12}\right) = \frac{|w|}{\sqrt{2}}$$

$$\frac{1}{2} = \frac{|w|}{\sqrt{2}}$$

$$\Rightarrow |w| = \frac{1}{\sqrt{2}}$$

2biv)

$$w^6 = \left|\frac{1}{\sqrt{2}}\right|^6 e^{j\left(\frac{7\pi}{12} \times 6\right)}$$

$$= \frac{1}{8}\left(e^{j\frac{7\pi}{2}}\right)$$

$$= \frac{1}{8}\left(e^{-j\frac{\pi}{2}}\right)$$

$$= \frac{1}{8}\left(\cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{8}(0 - j)$$

$$= \frac{j}{8}$$

3) i) Characteristic eqn given by

$$\det(\underline{M} - \lambda \underline{I}_3) = 0$$

$$\begin{vmatrix} 3-\lambda & 5 & 2 \\ 5 & 3-\lambda & -2 \\ 2 & -2 & -4-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)((3-\lambda)(-4-\lambda)-4) - 5(5(-4-\lambda)+4) + 2(-10-2(3-\lambda)) = 0$$

$$(3-\lambda)(-16+\lambda+\lambda^2) - 5(-16-5\lambda) + 2(-16+2\lambda) = 0$$

$$-48 + 16\lambda + 3\lambda - \lambda^2 + 3\lambda^2 - \lambda^3 + 80 + 25\lambda - 32 + 4\lambda = 0$$

$$-\lambda^3 + 2\lambda^2 + 48\lambda = 0$$

$$\lambda^3 - 2\lambda^2 - 48\lambda = 0$$

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3 ii)

$$\lambda^3 - 2\lambda^2 - 48\lambda = 0$$

$$\lambda(\lambda^2 - 2\lambda - 48) = 0$$

$$\lambda(\lambda-8)(\lambda+6) = 0$$

$\Rightarrow \lambda = 0$
or $\lambda = 8$
or $\lambda = -6$

$$\begin{pmatrix} 3 & 5 & 2 \\ 5 & 3 & -2 \\ 2 & -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$3x + 5y + 2z = 8x$ ①
$5x + 3y - 2z = 8y$ ②
$2x - 2y - 4z = 8z$ ③

① + ②

$$8x + 8y = 8x + 8y$$

No help!

5③ + 2①

$$16x - 16z = 40z + 16x$$
$$0 = 56z$$
$$\Rightarrow z = 0$$

Subst in ①

$$3x + 5y + 0 = 8x$$
$$5y = 5x$$
$$y = x$$

So for $\lambda = 8$ choose eigenvector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 3 & 5 & 2 \\ 5 & 3 & -2 \\ 2 & -2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$3x + 5y + 2z = -6x$ ①
$5x + 3y - 2z = -6y$ ②
$2x - 2y - 4z = -6z$ ③

①+② $8x + 8y = -6x - 6y$
$14x = -14y$

3ii) cont)

$$y = \frac{14z}{-14} = -x$$

Sub in ①

$$3x - 5x + 2z = -6x$$
$$2z = -4x$$
$$z = -2x$$

So for $\lambda = -6$ choose

eigenvector $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

3iii)

$$\underline{P} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}$$

$$\underline{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -6 \end{pmatrix}^2$$

$$\underline{D} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 36 \end{pmatrix}$$

3iv)

$$\underline{M}^4 = a\underline{M}^2 + b\underline{M}$$

Cayley - Hamilton Theorem

$\underline{M}$ satisfies its own

characteristic equation

so $\underline{M}^3 - 2\underline{M}^2 - 48\underline{M} = \underline{0}$ (*)

Post-multiplying by $\underline{M}$

$$\underline{M}^4 - 2\underline{M}^3 - 48\underline{M}^2 = \underline{0}$$

$$\therefore \quad \underline{M}^4 = 2\underline{M}^3 + 48\underline{M}^2$$

But from (*) $\underline{M}^3 = 2\underline{M}^2 + 48\underline{M}$

so $\underline{M}^4 = 2(2\underline{M}^2 + 48\underline{M}) + 48\underline{M}^2$

$$\underline{M}^4 = 4\underline{M}^2 + 96\underline{M} + 48\underline{M}^2$$

$$\underline{M}^4 = 52\underline{M}^2 + 96\underline{M}$$

4 a)
$$\int_0^1 \frac{1}{\sqrt{9x^2+16}}\,dx$$
$$= \frac{1}{3}\int_0^1 \frac{1}{\sqrt{x^2+\left(\frac{4}{3}\right)^2}}\,dx$$
$$= \frac{1}{3}\left[\operatorname{arsinh}\left(\frac{x}{4/3}\right)\right]_0^1$$
$$= \frac{1}{3}\left[\ln\left(x+\sqrt{x^2+\tfrac{16}{9}}\right)\right]_0^1$$
$$= \frac{1}{3}\left(\ln\left(1+\tfrac{5}{3}\right)-\ln\left(0+\tfrac{4}{3}\right)\right)$$
$$= \frac{1}{3}\left(\ln\left(8/3\right)-\ln\left(\tfrac{4}{3}\right)\right)$$
$$= \frac{1}{3}\left(\ln\left(\frac{8/3}{4/3}\right)\right) = \frac{1}{3}\ln 2$$

4 b i)
$$2\sinh x\cosh x$$
$$= 2\left(\tfrac{1}{2}(e^x-e^{-x})\right)\left(\tfrac{1}{2}(e^x+e^{-x})\right)$$
$$= \frac{1}{2}\left((e^x-e^{-x})(e^x+e^{-x})\right)$$
$$= \frac{1}{2}\left[e^{2x}-1+1-e^{-2x}\right]$$
$$= \frac{1}{2}\left(e^{2x}-e^{-2x}\right)$$
$$= \sinh(2x)$$

4 b ii)
$$y = 20\cosh x - 3\cosh 2x$$
$$\frac{dy}{dx} = 20\sinh x - 6\sinh 2x$$
$$\frac{dy}{dx} = 20\sinh x - 12\sinh x\cosh x$$

At st. pt $\frac{dy}{dx}=0$

$\Rightarrow 20\sinh x - 12\sinh x\cosh x = 0$

$\Rightarrow 4\sinh x(5-3\cosh x) = 0$

$\Rightarrow \sinh x = 0$

or $\cosh x = \frac{5}{3}$

When $\sinh x = 0$ $\quad x = 0$

$$y = 20\cosh 0 - 3\cosh 0$$
$$= 20 - 3$$
$$= 17$$

St pt at $(0, 17)$

When $\cosh x = \frac{5}{3}$

$$x = \operatorname{arcosh} 5/3$$
$$x = \ln\left(5/3 + \sqrt{\tfrac{25}{9}-1}\right)$$
$$x = \ln\left(5/3 + 4/3\right) = \ln 3$$

Also $x = -\ln 3$ due to symmetry of $y=\cosh x$

When $x = \ln 3$

$$y = 20\times\tfrac{5}{3} - 3\left(\tfrac{1}{2}\left(e^{2\ln 3}+e^{-2\ln 3}\right)\right)$$

4ii) cont

$y = \frac{100}{3} - 3\left(\frac{1}{2}\left(e^{\ln 9} + e^{\ln \frac{1}{9}}\right)\right)$

$y = \frac{100}{3} - \frac{3}{2}\left(9 + \frac{1}{9}\right)$

$y = \frac{100}{3} - \frac{3}{2} \times \frac{82}{9}$

$y = \frac{100}{3} - \frac{41}{3} = \frac{59}{3}$

st pt at $\left(\ln 3, \frac{59}{3}\right)$

When $x = -\ln 3$

$y = 20 \times \frac{5}{3} - 3\left(\frac{1}{2}\left(e^{-2\ln 3} + e^{2\ln 3}\right)\right)$

$y = \frac{59}{3}$ as before

st pt at $\left(-\ln 3, \frac{59}{3}\right)$

4iii) $\int_{-\ln 3}^{\ln 3} \left(20\cosh x - 3\cosh 2x\right) dx$

$= \left[20\sinh x - \frac{3}{2}\sinh 2x\right]_{-\ln 3}^{\ln 3}$

$= \left(\frac{20}{2}\left(e^{\ln 3} - e^{-\ln 3}\right) - \frac{3}{4}\left(e^{2\ln 3} - e^{-2\ln 3}\right)\right)$

$- \left(\frac{20}{2}\left(e^{-\ln 3} - e^{\ln 3}\right) - \frac{3}{4}\left(e^{-2\ln 3} - e^{2\ln 3}\right)\right)$

$= \left(10\left(3 - \frac{1}{3}\right) - \frac{3}{4}\left(9 - \frac{1}{9}\right)\right)$

$- \left(10\left(\frac{1}{3} - 3\right) - \frac{3}{4}\left(\frac{1}{9} - 9\right)\right)$

$= \left(\frac{80}{3} - \frac{20}{3}\right) - \left(-\frac{80}{3} + \frac{20}{3}\right)$

$= 20 - (-20)$

$= 40$ as required