1) a) i)

$$f(t) = \arcsin(t)$$

$$f'(t) = \frac{1}{\sqrt{1-t^2}}$$

$$\Rightarrow f'(t) = (1-t^2)^{-\frac{1}{2}}$$

$$f''(t) = -\tfrac{1}{2}(1-t^2)^{-3/2}(-2t)$$

$$f''(t) = \frac{t}{(1-t^2)^{3/2}}$$

ii) Let $f(x) = \arcsin x$

Then $f(a+x)$

$$\approx f(a) + xf'(a) + \frac{x^2}{2!}f''(a)$$

$$f(\tfrac{1}{2}+x) = \arcsin(x+\tfrac{1}{2})$$

$$\approx \arcsin(\tfrac{1}{2}) + x\frac{1}{\sqrt{1-(\frac{1}{2})^2}} + \frac{x^2}{2}\left(\frac{\frac{1}{2}}{(1-(\frac{1}{2})^2)^{3/2}}\right)$$

$$= \frac{\pi}{6} + \frac{x}{\sqrt{3/4}} + \frac{x^2}{2}\left(\frac{\frac{1}{2}}{(\sqrt{3/4})^3}\right)$$

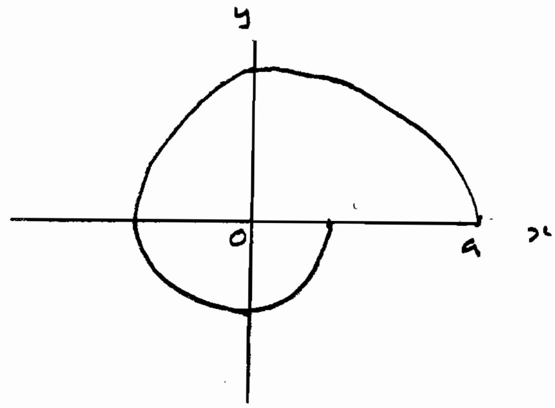
$$= \frac{\pi}{6} + \frac{2x}{\sqrt{3}} + \frac{x^2}{4}\left(\frac{8}{3\sqrt{3}}\right)$$

$$= \frac{\pi}{6} + \frac{2}{\sqrt{3}}x + \frac{2x^2}{3\sqrt{3}}$$

b)

$$r = \frac{\pi a}{\pi + \theta}$$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	a	$\frac{4a}{5}$	$\frac{2a}{3}$	$\frac{4a}{7}$	$\frac{a}{2}$	$\frac{4a}{9}$	$\frac{2a}{5}$	$\frac{4a}{11}$	$\frac{a}{3}$



c)

$$\int_0^{3/2} \frac{1}{9+4x^2}\,dx$$

$$= \int_0^{3/2} \frac{1}{4(9/4+x^2)}\,dx$$

$$= \frac{1}{4}\left[\frac{1}{3/2}\tan^{-1}\left(\frac{x}{3/2}\right)\right]_0^{3/2}$$

$$= \frac{1}{4}\left[\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right]_0^{3/2}$$

$$= \frac{1}{4}\times\frac{2}{3}\left[\tan^{-1}1 - \tan^{-1}0\right]$$

$$= \frac{1}{6}\left[\frac{\pi}{4} - 0\right]$$

$$= \frac{\pi}{24}$$

2) a)

$$z = \cos\theta + j\sin\theta$$

$$z^n = \cos n\theta + j\sin n\theta$$

$$z^{-n} = \cos n\theta - j\sin n\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z^n - \frac{1}{z^n} = 2j\sin n\theta$$

$$z - \frac{1}{z} = 2j\sin\theta$$

$$(2j\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$$

$$= z^5 - 5z^4\frac{1}{z} + 10z^3\frac{1}{z^2} - 10z^2\frac{1}{z^3} + 5z\frac{1}{z^4} - \frac{1}{z^5}$$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$= 2j\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]$$

$$\therefore 32j\sin^5\theta = 2j\left[\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right]$$

$$\Rightarrow \sin^5\theta = \frac{\sin 5\theta - 5\sin 3\theta + 10\sin\theta}{16}$$

$A = \frac{10}{16}$ $\quad B = \frac{-5}{16}$ $\quad C = \frac{1}{16}$

2b) i)

$$-9j = 9e^{-j\frac{\pi}{2}}$$

Modulus of 4th roots

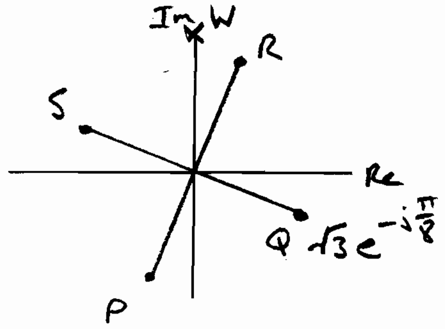
$$= \sqrt[4]{9} = \sqrt{3}$$

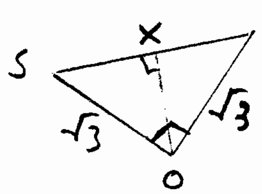
arguments of roots

$$-\frac{\pi}{8} + \frac{2n\pi}{4} \quad \text{for } n = 0, 1, 2, 3$$

$$= -\frac{\pi}{8},\ \frac{3\pi}{8},\ \frac{7\pi}{8},\ \cancel{\frac{11\pi}{8}}\ -\frac{5\pi}{8}$$

$$\sqrt{3}e^{-j\frac{\pi}{8}} \quad \sqrt{3}e^{j\frac{3\pi}{8}} \quad \sqrt{3}e^{j\frac{7\pi}{8}} \quad \sqrt{3}e^{-j\frac{5\pi}{8}}$$





Find $|OX|$

$$\sin\frac{\pi}{4} = \frac{|OX|}{\sqrt{3}}$$

$$\frac{1}{\sqrt{2}} = \frac{|OX|}{\sqrt{3}}$$

$$\Rightarrow |OX| = \frac{\sqrt{3}}{\sqrt{2}}$$

$$\arg X = \arg R + \frac{\pi}{4} = \frac{3\pi}{8} + \frac{\pi}{4} = \frac{5\pi}{8}$$

$$\therefore |w| = \left(\frac{\sqrt{3}}{\sqrt{2}}\right)^4 = \frac{9}{4}$$

$$\arg w = \frac{5\pi}{8} \times 4 = \frac{20\pi}{8} = \frac{\pi}{2}$$

$$\therefore w = \frac{9}{4}j$$

3) $2\lambda^3+\lambda^2-13\lambda+6=0$

i) $2(2)^3+(2)^2-13(2)+6$

$= 16+4-26+6=0$

$\therefore$ $\lambda=2$ is eigen value

$$\begin{array}{r|l} & 2\lambda^2+5\lambda-3 \\ \hline \lambda-2 & 2\lambda^3+\lambda^2-13\lambda+6 \\ & \underline{2\lambda^3-4\lambda^2} \\ & \quad 5\lambda^2-13\lambda \\ & \quad \underline{5\lambda^2-10\lambda} \\ & \qquad -3\lambda+6 \\ & \qquad \underline{-3\lambda+6} \end{array}$$

$(\lambda-2)(2\lambda^2+5\lambda-3)=0$

$(\lambda-2)(2\lambda-1)(\lambda+3)=0$

$\Rightarrow$ $\lambda=2$, $\lambda=\frac{1}{2}$, $\lambda=-3$

---

ii) $\underline{M}\begin{pmatrix}3\\-3\\1\end{pmatrix} = 2\begin{pmatrix}3\\-3\\1\end{pmatrix} = \begin{pmatrix}6\\-6\\2\end{pmatrix}$

$\underline{M}^2\begin{pmatrix}1\\-1\\\frac{1}{3}\end{pmatrix} = \frac{1}{3}\underline{M}^2\begin{pmatrix}3\\-3\\1\end{pmatrix}$

$= \frac{1}{3}\underline{M}\,\underline{M}\begin{pmatrix}3\\-3\\1\end{pmatrix}$

$= \frac{1}{3}\underline{M}\begin{pmatrix}6\\-6\\2\end{pmatrix}$

$= \frac{2}{3}\underline{M}\begin{pmatrix}3\\-3\\1\end{pmatrix}$

$= \frac{2}{3}\begin{pmatrix}6\\-6\\2\end{pmatrix} = \begin{pmatrix}4\\-4\\4/3\end{pmatrix}$

$\underline{M}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\-3\\1\end{pmatrix}$

Now $\underline{M}\begin{pmatrix}3\\-3\\1\end{pmatrix} = 2\begin{pmatrix}3\\-3\\1\end{pmatrix}$

$\therefore$ $\underline{M}\begin{pmatrix}3/2\\-3/2\\1/2\end{pmatrix} = \begin{pmatrix}3\\-3\\1\end{pmatrix}$

$\therefore$ $x=\frac{3}{2}$, $y=-\frac{3}{2}$, $z=\frac{1}{2}$

---

iii) Cayley - Hamilton Theorem

$2\underline{M}^3+\underline{M}^2-13\underline{M}+6\underline{I}=\underline{0}$

$\Rightarrow 2\underline{M}^4+\underline{M}^3-13\underline{M}^2+6\underline{M}=0$

$\Rightarrow 2\underline{M}^4=-\underline{M}^3+13\underline{M}^2-6\underline{M}$

$\Rightarrow 2\underline{M}^4=-\left(\frac{1}{2}(-\underline{M}^2+13\underline{M}-6\underline{I}\right) + 13\underline{M}^2-6\underline{M}$

$\Rightarrow 2\underline{M}^4=\frac{27}{2}\underline{M}^2-\frac{25}{2}\underline{M}+3\underline{I}$

$\Rightarrow \underline{M}^4=\frac{27}{4}\underline{M}^2-\frac{25}{4}\underline{M}+\frac{3}{2}\underline{I}$

4) i) $2\sinh x\cosh x$

$= 2 \times \frac{1}{2}(e^{x} - e^{-x}) \times \frac{1}{2}(e^{x} + e^{-x})$

$= \frac{1}{2}\left(e^{2x} - 1 + 1 - e^{-2x}\right)$

$= \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$

$= \sinh 2x$

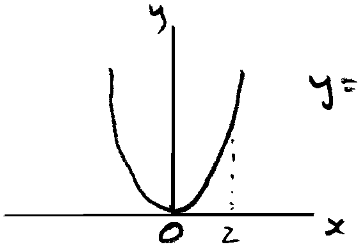
$\therefore \sinh 2x = 2\sinh x\cosh x$

Differentiating

$2\cosh 2x = 2\sinh x \sinh x + \cosh x(2\cosh x)$

$\Rightarrow \cosh 2x = \sinh^2 x + \cosh^2 x$

ii



$y = \cosh x - 1$

$V = \pi \int_0^2 y^2\,dx$

$V = \pi \int_0^2 (\cosh x - 1)^2\,dx$

$= \pi \int_0^2 \left(\cosh^2 x - 2\cosh x + 1\right)dx$

$= \pi \int_0^2 \left(\frac{1 + \cosh 2x}{2} - 2\cosh x + 1\right)dx$

$= \pi \left[\frac{x}{2} + \frac{\sinh 2x}{4} - \sinh x + x\right]_0^2$

$= \pi\left[\left(1 + \frac{\sinh 4}{4} - \sinh 2 + 2\right) - \left(0 + 0 - 0 + 0\right)\right]$

$= 19.464$

iii) $y = \cosh 2x + \sinh x$

$\frac{dy}{dx} = 2\sinh 2x + \cosh x$

At st. pt. $\frac{dy}{dx} = 0$

$\Rightarrow \quad 2\sinh 2x + \cosh x = 0$

$4\sinh x\cosh x + \cosh x = 0$

$\cosh x\,(4\sinh x + 1) = 0$

$\cosh x \neq 0 \ \forall x$

$\Rightarrow$ only root when

$4\sinh x + 1 = 0$

$x = \text{arsinh}\left(-\frac{1}{4}\right)$

$x = \ln\left(-\frac{1}{4} + \sqrt{\left(-\frac{1}{4}\right)^2 + 1}\right)$

$x = \ln\left(-\frac{1}{4} + \sqrt{\frac{17}{16}}\right)$

$x = \ln\left(\frac{\sqrt{17} - 1}{4}\right)$