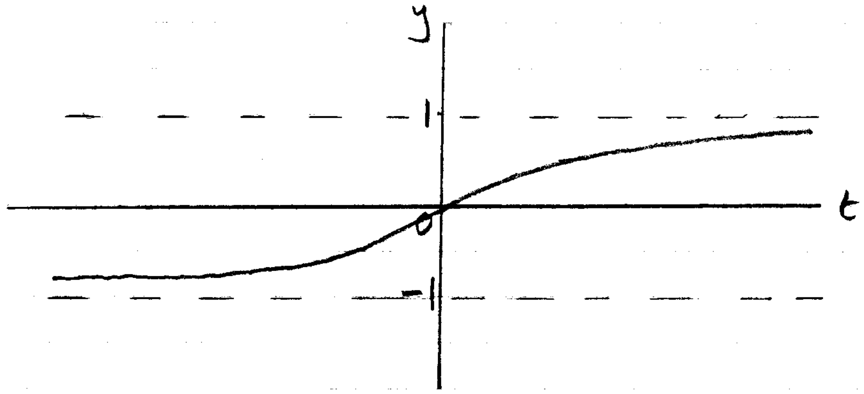
FP2 JANUARY 2012 Q4 HYPERBOLIC FUNCTIONS

4) i) $\tanh t = \dfrac{\sinh t}{\cosh t} = \dfrac{\frac{1}{2}(e^t - e^{-t})}{\frac{1}{2}(e^t + e^{-t})}$

$$\tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}}$$



$y = \tanh t$

ii) Let $y = \operatorname{artanh} x$

$\Rightarrow \quad \tanh y = x$

$\Rightarrow \quad \dfrac{e^y - e^{-y}}{e^y + e^{-y}} = x$

$\Rightarrow \quad e^y - e^{-y} = xe^y + xe^{-y}$

Multiplying by e^y

$\Rightarrow \quad e^{2y} - 1 = xe^{2y} + x$

$\Rightarrow \quad e^{2y} - xe^{2y} = x + 1$

$\Rightarrow \quad (1-x)e^{2y} = 1 + x$

$\Rightarrow \quad e^{2y} = \dfrac{1+x}{1-x}$

$\Rightarrow \quad 2y = \ln\left(\dfrac{1+x}{1-x}\right)$

4 ii) cont

$$\Rightarrow \quad y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\Rightarrow \quad \operatorname{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Valid for $-1 < x < 1$

iii)

$$\tanh y = x \qquad (\Rightarrow y = \operatorname{artanh} x)$$

d wrt x

$$\operatorname{sech}^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1-\tanh^2 y}$$

$$\frac{dy}{dx} = \frac{1}{1-x^2}$$

$$y = \operatorname{artanh} x$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\ln(1+x) - \frac{1}{2}\ln(1-x)$$

$$\frac{dy}{dx} = \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

$$\frac{dy}{dx} = \frac{(1-x)+(1+x)}{2(1+x)(1-x)}$$

4 iii cont)

$$\frac{dy}{dx} = \frac{2}{2(1-x^2)} = \frac{1}{1-x^2}$$

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iv)

$$\int_0^{\frac{1}{2}} \text{artanh}\,x \, dx = \int_0^{\frac{1}{2}} 1 \times \text{artanh}\,x \, dx$$

Let $u = \text{artanh}\,x$ Let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = \frac{1}{1-x^2}$ $\Rightarrow v = x$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$$\int_0^{\frac{1}{2}} \text{artanh}\,x = \Big[x \,\text{artanh}\,x\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{1-x^2} \, dx$$

$$= \Big[x \,\text{artanh}\,x\Big]_0^{\frac{1}{2}} + \frac{1}{2}\Big[\ln(1-x^2)\Big]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[x \ln\left(\frac{1+x}{1-x}\right)\right]_0^{\frac{1}{2}} + \frac{1}{2}\Big[\ln(1-x^2)\Big]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[\frac{1}{2}\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) - 0 + \ln\left(\frac{3}{4}\right) - \ln 1\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\ln 3 + \ln\frac{3}{4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\ln 3 + \frac{1}{2}\ln\left(\frac{3}{4}\right)^2\right]$$

$$= \frac{1}{4}\left[\ln 3 + \ln\left(\frac{9}{16}\right)\right] = \frac{1}{4}\ln\left(\frac{27}{16}\right)$$