FP2 JANUARY 2006 Q4 HYPERBOLIC FUNCTIONS

4a)

$$\sinh x + 4\cosh x = 8$$

$$\tfrac{1}{2}(e^{x} - e^{-x}) + 2(e^{x} + e^{-x}) = 8$$

$$e^{x} - e^{-x} + 4e^{x} + 4e^{-x} = 16$$

$$5e^{x} + 3e^{-x} = 16$$

$$5e^{2x} + 3 = 16e^{x} \qquad (\times e^{x})$$

$$5e^{2x} - 16e^{x} + 3 = 0$$

$$(5e^{x} - 1)(e^{x} - 3) = 0$$

$$\Rightarrow 5e^{x} - 1 = 0 \quad \text{or} \quad e^{x} - 3 = 0$$

$$5e^{x} = 1 \qquad e^{x} = 3$$

$$e^{x} = \tfrac{1}{5} \qquad x = \ln 3$$

$$x = \ln\left(\tfrac{1}{5}\right)$$

$$x = -\ln 5$$

4b)

$$\int_0^2 e^{x} \sinh x \, dx$$

$$= \int_0^2 e^{x} \times \tfrac{1}{2}(e^{x} - e^{-x}) \, dx$$

$$= \frac{1}{2}\int_0^2 (e^{2x} - 1) \, dx$$

$$= \frac{1}{2}\left[\frac{1}{2}e^{2x} - x\right]_0^2$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}e^{4} - 2\right) - \left(\frac{1}{2}e^{0} - 0\right)\right]$$

4b) cont

$$= \frac{1}{2}\left[\frac{1}{2}e^4 - 2 - \frac{1}{2}\right]$$

$$= \frac{1}{4}e^4 - \frac{5}{4}$$

$$= \frac{1}{4}\left(e^4 - 5\right)$$

4c) From first principles

i)

Let $y = \operatorname{arsinh}\left(\frac{2x}{3}\right)$

$\Rightarrow \sinh y = \frac{2x}{3}$

$\cosh y \frac{dy}{dx} = \frac{2}{3}$ d.w.r.t.x

$$\frac{dy}{dx} = \frac{2}{3\cosh y}$$

$$\frac{dy}{dx} = \frac{2}{3(\sqrt{1+\sinh^2 y})}$$

$$\frac{dy}{dx} = \frac{2}{3\sqrt{1+\frac{4x^2}{9}}}$$

$$\frac{dy}{dx} = \frac{2}{\sqrt{9+4x^2}}$$

$$\therefore \frac{d}{dx}\operatorname{arsinh}\left(\frac{2x}{3}\right) = \frac{2}{\sqrt{9+4x^2}}$$

4c)
i) Same question using standard differential obtained from tables

$$\frac{d}{dx} \operatorname{arsinh} x = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{d}{dx} \operatorname{arsinh}\left(\frac{2x}{3}\right)$$

Let $u = \frac{2x}{3}$

$$\frac{du}{dx} = \frac{2}{3}$$

$$\frac{d}{dx} \operatorname{arsinh}\left(\frac{2x}{3}\right) = \frac{d}{du} \operatorname{arsinh} u \times \frac{du}{dx}$$ chain rule

$$= \frac{1}{\sqrt{1+u^2}} \times \frac{2}{3}$$

$$= \frac{2}{3\sqrt{1+\left(\frac{2x}{3}\right)^2}}$$

$$= \frac{2}{\sqrt{9+4x^2}}$$ as before.

4c)
ii) $$\int_0^2 \operatorname{arsinh}\left(\frac{2x}{3}\right) dx$$

Let $u = \operatorname{arsinh}\left(\frac{2x}{3}\right)$ Let $\frac{dv}{dx} = 1$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{9+4x^2}} \qquad \Rightarrow v = x$$

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4c ii) cont

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$$\int_0^2 \operatorname{arsinh}\left(\frac{2x}{3}\right) dx = \left[x \operatorname{arsinh}\left(\frac{2x}{3}\right)\right]_0^2 - \int_0^2 \frac{2x}{\sqrt{9+4x^2}}\, dx$$

First evaluate $\int_0^2 \frac{2x}{\sqrt{9+4x^2}}\, dx$

Let $u = 9 + 4x^2$

$\frac{du}{dx} = 8x$

When $x = 2$, $u = 25$
$x = 0$, $u = 9$

$$= \int_9^{25} \frac{1}{4} \frac{1}{\sqrt{u}}\, du$$

$$= \int_9^{25} \frac{u^{-\frac{1}{2}}}{4}\, du$$

$$= \left[\frac{u^{\frac{1}{2}}}{4 \times \frac{1}{2}}\right]_9^{25} = \left[\frac{\sqrt{u}}{2}\right]_9^{25} = \frac{5}{2} - \frac{3}{2} = 1$$

$$\therefore \int_0^2 \operatorname{arsinh}\left(\frac{2x}{3}\right) = \left[x \operatorname{arsinh}\left(\frac{2x}{3}\right)\right]_0^2 - 1$$

$$= \left(2 \operatorname{arsinh}\left(\frac{4}{3}\right) - 0\right) - 1$$

$$= 2 \ln\left(\frac{4}{3} + \sqrt{\left(\frac{4}{3}\right)^2 + 1}\right) - 1$$

$$= 2 \ln\left(\frac{4}{3} + \sqrt{\frac{16+9}{9}}\right) - 1$$

$$= 2 \ln\left(\frac{4}{3} + \frac{5}{3}\right) - 1$$

$$= 2 \ln 3 - 1$$