FP2 JANUARY 2009 Q4 HYPERBOLIC FUNCTIONS

4 a) i)

$$\cosh x = \tfrac{1}{2}(e^{x}+e^{-x}), \quad \sinh x = \tfrac{1}{2}(e^{x}-e^{-x})$$

$$\cosh^2 x - \sinh^2 x = \tfrac{1}{4}(e^{x}+e^{-x})^2 - \tfrac{1}{4}(e^{x}-e^{-x})^2$$

$$= \tfrac{1}{4}\left[e^{2x}+2+e^{-2x} - (e^{2x}-2+e^{-2x})\right]$$

$$= \tfrac{1}{4}\left[e^{2x}+2+e^{-2x} - e^{2x}+2-e^{-2x}\right]$$

$$= \tfrac{1}{4}[4] = 1$$

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ii) Given $\sinh x = \tan y \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$

A)

$$\Rightarrow \sinh^2 x = \tan^2 y$$

$$\Rightarrow 1 + \sinh^2 x = 1 + \tan^2 y$$

$$\Rightarrow \cosh^2 x = \sec^2 y = \frac{1}{\cos^2 y}$$

$$\Rightarrow \cosh x = \frac{1}{\cos y}$$

$$\therefore \tanh x = \frac{\sinh x}{\cosh x} = \frac{\tan y}{\frac{1}{\cos y}} = \frac{\frac{\sin y}{\cos y}}{\frac{1}{\cos y}}$$

$$\Rightarrow \tanh x = \frac{\sin y}{\cos y} \times \frac{\cos y}{1} = \sin y$$

as required

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4 ii) B) Show $x = \ln(\tan y + \sec y)$

If $\sinh x = \tan y$

$$x = \operatorname{arsinh}(\tan y)$$
$$x = \ln\left(\tan y + \sqrt{1+\tan^2 y}\right)$$
$$x = \ln\left(\tan y + \sqrt{\sec^2 y}\right)$$
$$x = \ln(\tan y + \sec y) \quad \text{as required}$$

4b) Given $y = \operatorname{artanh} x$

i) $\Rightarrow \tanh y = x$

$\Rightarrow \operatorname{sech}^2 y \dfrac{dy}{dx} = 1$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1-\tanh^2 y}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1-x^2}$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\,dx = \left[\operatorname{artanh} x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$
$$= \operatorname{artanh}\left(\tfrac{1}{2}\right) - \operatorname{artanh}\left(-\tfrac{1}{2}\right)$$
$$= 2\operatorname{artanh}\left(\tfrac{1}{2}\right)$$

since $\operatorname{artanh} x$ is an odd function

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4b ii) $$\frac{1}{1-x^2} \equiv \frac{1}{(1+x)(1-x)} \equiv \frac{A}{1+x} + \frac{B}{1-x}$$

$$1 \equiv A(1-x) + B(1+x)$$

$x=1$ gives $1 = B(1+1) \Rightarrow B = \frac{1}{2}$

$x=-1$ gives $1 = A(1--1) \Rightarrow A = \frac{1}{2}$

$$\frac{1}{1-x^2} = \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$$

$$\therefore \int \frac{1}{1-x^2}\,dx = \int\left(\frac{1}{2(1+x)} + \frac{1}{2(1-x)}\right)dx$$

$$= \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x| + c$$

$$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + c$$

iii) $$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1-x^2}\,dx = \frac{1}{2}\left[\ln\left|\frac{1+x}{1-x}\right|\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{2}\left[\ln\left(\frac{\frac{3}{2}}{\frac{1}{2}}\right) - \ln\left(\frac{\frac{1}{2}}{3/2}\right)\right]$$

$$= \frac{1}{2}\left[\ln 3 - \ln\frac{1}{3}\right]$$

$$= \frac{1}{2}\left[\ln 3 + \ln 3\right]$$

$$= \ln 3$$

From part(i) and part(ii) $2\,\text{artanh}(\frac{1}{2}) = \ln 3$ so $\text{artanh}(\frac{1}{2}) = \frac{1}{2}\ln 3$

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