MEI CORE 3 JANUARY 2006

1) $y = (1+6x)^{\frac{1}{3}}$

$$\frac{dy}{dx} = \frac{1}{3}(1+6x)^{-\frac{2}{3}} \times 6$$

$$= \frac{2}{(1+6x)^{2/3}}$$

$$= \frac{2}{y^2}$$

2) $P = 5 + ae^{-bt}$

i) At $t=0$, $P=8$

At $t=1$, $P=6$

$\therefore$ $8 = 5 + ae^0$

$8 = 5 + a$

$\Rightarrow$ $a = 3$

Also $6 = 5 + 3e^{-b\times 1}$

$1 = 3e^{-b}$

$\frac{1}{3} = e^{-b}$

$\ln\left(\frac{1}{3}\right) = -b$

$\Rightarrow b = -\ln\left(\frac{1}{3}\right)$

$b = 1.0986$

$b = 1.10$ to 3 s.f.

2 ii)

Long term $P \to 5 + 0$

Population 5 million

3 i) $2\ln x + \ln 3$

$= \ln x^2 + \ln 3$

$= \ln(3x^2)$

3 ii) $2\ln x + \ln 3 = \ln(5x+2)$

$\Rightarrow$ $\ln(3x^2) = \ln(5x+2)$

$\Rightarrow$ $3x^2 = 5x+2$

$\Rightarrow$ $3x^2 - 5x - 2 = 0$

3 iii) $3x^2 - 5x - 2 = 0$

$(3x+1)(x-2) = 0$

$\Rightarrow$ $x = -\frac{1}{3}$ or $x = 2$

$\ln x$ defined only for $x > 0$

$\therefore$ $x = 2$ is only valid root for this eqn.

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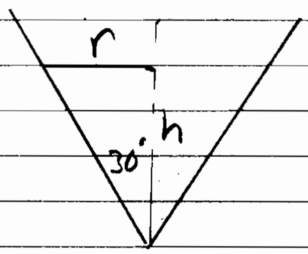
4 i) Given $\frac{dV}{dt} = 2 \text{ cm}^3 \text{s}^{-1}$

4 ii)



$$V = \frac{1}{3}\pi r^2 h$$

but $\tan 30° = \frac{r}{h}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{r}{h}$$

$$\Rightarrow h = \sqrt{3}\, r$$

$$\therefore V = \frac{1}{3}\pi r^2 \times \sqrt{3}\, r$$

$$V = \frac{\sqrt{3}\pi r^3}{3}$$

$$\frac{dV}{dr} = \frac{3\sqrt{3}\pi r^2}{3}$$

$$\frac{dV}{dr} = \sqrt{3}\pi r^2$$

4 iii)

$$\frac{dr}{dt} = \frac{dr}{dV}\frac{dV}{dt}$$

$$= \frac{\frac{dV}{dt}}{\frac{dV}{dr}}$$

$$\Rightarrow \frac{dr}{dt} = \frac{2}{\sqrt{3}\pi r^2} \text{ cm s}^{-1}$$

when $r = 2$ cm

$$\frac{dr}{dt} = \frac{2}{\sqrt{3}\pi \times 4}$$

$$= \frac{1}{2\sqrt{3}\pi} \text{ cm s}^{-1}$$

$$= 0.0919 \text{ cm s}^{-1}$$

to 3 s.f.

5 i) $y^3 = 2xy + x^2$

$$\Rightarrow 3y^2\frac{dy}{dx} = 2x\frac{dy}{dx} + 2y + 2x$$

$$3y^2\frac{dy}{dx} - 2x\frac{dy}{dx} = 2x + 2y$$

$$(3y^2 - 2x)\frac{dy}{dx} = 2(x+y)$$

$$\frac{dy}{dx} = \frac{2(x+y)}{(3y^2-2x)}$$

5 ii)

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$= \frac{3y^2 - 2x}{2(x+y)}$$

6i) $f(x) = 1 + 2\sin x$

for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

Let

$y = 1 + 2\sin x$

Swap variables and make y the subject

$x = 1 + 2\sin y$

$x - 1 = 2\sin y$

$\frac{x-1}{2} = \sin y$

$\arcsin\left(\frac{x-1}{2}\right) = y$

$\therefore f^{-1}(x) = \arcsin\left(\frac{x-1}{2}\right)$

Domain of $f^{-1}(x)$ is

given by $-1 \le x \le 3$

6 ii)

$A\left(\frac{\pi}{2}, 3\right)$

$B(1, 0)$

$C\left(3, \frac{\pi}{2}\right)$

7)i) $y = 2x - x\ln x$ for $x > 0$

At A, $0 = 2x - x\ln x$

$0 = x(2 - \ln x)$

$\Rightarrow x = 0$ or $\ln x = 2$

At A, $\ln x = 2$

$x = e^2$

7ii)

$y = 2x - x\ln x$

$\frac{dy}{dx} = 2 - x\frac{1}{x} - \ln x$

$= 2 - 1 - \ln x$

$\frac{dy}{dx} = 1 - \ln x$

At t.p. B, $\frac{dy}{dx} = 0$

$\Rightarrow 1 - \ln x = 0$

$\Rightarrow \ln x = 1$

$\Rightarrow x = e$

When $x = e$

$y = 2e - e\ln e$

$= 2e - e$

$= e$

$\therefore$ B is point (e, e)

7iii) At $A(e^2, 0)$

$\frac{dy}{dx} = 1 - \ln x$

$= 1 - \ln e^2$

$= 1 - 2\ln e$

$= 1 - 2 = -1$

At C, $x = 1$

$\therefore \frac{dy}{dx} = 1 - \ln x$

$= 1 - \ln 1$

$= 1 - 0$

$= 1$

tgts are $\perp$ since

$-1 \times 1 = -1$

7iv)

$\int x \ln x \, dx$

Let $u = \ln x$, Let $\frac{dv}{dx} = x$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\Rightarrow v = \frac{x^2}{2}$

$\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int x \ln x \, dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x} dx$

$= \frac{x^2}{2}\ln x - \int \frac{x}{2} dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

$$\text{Area} = \int_1^e (2x - x\ln x)\, dx$$

$$= \left[x^2 - \left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\right]_1^e$$

$$= \left[\frac{5x^2}{4} - \frac{x^2}{2}\ln x\right]_1^e$$

$$= \frac{1}{4}\left[5x^2 - 2x^2\ln x\right]_1^e$$

$$= \frac{1}{4}\left[(5e^2 - 2e^2\ln e) - (5 - 2\ln 1)\right]$$

$$= \frac{1}{4}\left[3e^2 - 5\right]$$

$$= \frac{3e^2}{4} - \frac{5}{4}$$

8) $f(x) = \dfrac{\sin x}{2 - \cos x}$

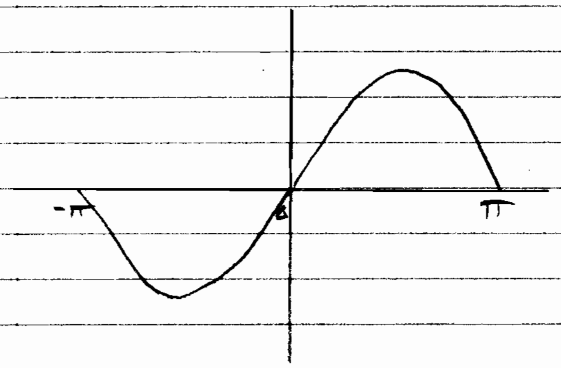
for $-\pi \le x \le \pi$

i) $f(-x) = \dfrac{\sin(-x)}{2 - \cos(-x)}$

$= \dfrac{-\sin x}{2 - \cos x}$

$= -\dfrac{\sin x}{2 - \cos x} = -f(x)$

$\therefore$ an odd function



At t.p $f'(x) = 0$

$\Rightarrow 2\cos x - 1 = 0$

$\Rightarrow \cos x = \frac{1}{2}$

$\Rightarrow x = \frac{\pi}{3}$

when $x = \frac{\pi}{3}$, $y = \dfrac{\sin\frac{\pi}{3}}{2 - \cos\frac{\pi}{3}}$

$y = \dfrac{\frac{\sqrt{3}}{2}}{2 - \frac{1}{2}}$

$y = \dfrac{\frac{\sqrt{3}}{2}}{\frac{3}{2}}$

$y = \frac{\sqrt{3}}{2} \times \frac{2}{3} = \frac{1}{\sqrt{3}}$

P is point $\left(\frac{\pi}{3}, \frac{1}{\sqrt{3}}\right)$

Range $-\frac{1}{\sqrt{3}} \le f(x) \le \frac{1}{\sqrt{3}}$

8ii) Using $\dfrac{d}{dx}\dfrac{u}{v} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$f'(x) = \dfrac{(2 - \cos x)\cos x - \sin x \sin x}{(2 - \cos x)^2}$

$f'(x) = \dfrac{2\cos x - \cos^2 x - \sin^2 x}{(2 - \cos x)^2}$

$f'(x) = \dfrac{2\cos x - 1(\cos^2 x + \sin^2 x)}{(2 - \cos x)^2}$

$f'(x) = \dfrac{2\cos x - 1}{(2 - \cos x)^2}$

8iii) $\displaystyle\int_0^\pi \frac{\sin x}{2 - \cos x}\,dx$

Since numerator is differential of denominator

$= \Big[\ln(2 - \cos x)\Big]_0^\pi$

$= \ln(2 - \cos\pi) - \ln(2 - \cos 0)$

$= \ln 3 - \ln 1$

$= \ln 3$

8iii) Alternative using substitution

$$\int_0^{\pi} \frac{\sin x}{2-\cos x}\,dx$$

Let $u = 2-\cos x$

$\Rightarrow \frac{du}{dx} = \sin x$

$\Rightarrow du = \sin x\,dx$

when $x = \pi$
$u = 2-\cos\pi = 3$

when $x = 0$
$u = 2-\cos 0 = 1$

Integral becomes

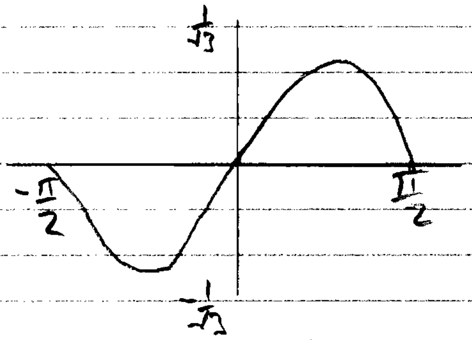
$$\int_1^3 \frac{1}{u}\,du$$

$= \Big[\ln u\Big]_1^3$

$= \ln 3 - \ln 1$

$= \ln 3$

iv) $f(2x)$



Stretch by s.f. $\frac{1}{2}$ parallel to x axis

v) $$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{2-\cos 2x}\,dx$$

$= \frac{1}{2}\ln 3$