OCR RECOGNISING ACHIEVEMENT

ADVANCED GCE **4753/01**

MATHEMATICS (MEI)

Methods for Advanced Mathematics (C3)

MONDAY 2 JUNE 2008

Morning

Time: 1 hour 30 minutes

Additional materials (enclosed): None

Additional materials (required):

Answer Booklet (8 pages)
Graph paper
MEI Examination Formulae and Tables (MF2)

INSTRUCTIONS TO CANDIDATES

- Write your name in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Read each question carefully and make sure you know what you have to do before starting your answer.
- Answer **all** the questions.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.
- The total number of marks for this paper is **72**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

This document consists of **4** printed pages.

Section A (36 marks)

1 Solve the inequality $|2x - 1| \leqslant 3$. [4]

2 Find $\int x e^{3x}\, dx$. [4]

3 **(i)** State the algebraic condition for the function $f(x)$ to be an even function.

What geometrical property does the graph of an even function have? [2]

(ii) State whether the following functions are odd, even or neither.

(A) $f(x) = x^2 - 3$

(B) $g(x) = \sin x + \cos x$

(C) $h(x) = \dfrac{1}{x + x^3}$ [3]

4 Show that $\displaystyle\int_1^4 \frac{x}{x^2 + 2}\, dx = \tfrac{1}{2}\ln 6$. [4]

5 Show that the curve $y = x^2 \ln x$ has a stationary point when $x = \dfrac{1}{\sqrt{e}}$. [6]

6 In a chemical reaction, the mass m grams of a chemical after t minutes is modelled by the equation

$$m = 20 + 30e^{-0.1t}.$$

(i) Find the initial mass of the chemical.

What is the mass of chemical in the long term? [3]

(ii) Find the time when the mass is 30 grams. [3]

(iii) Sketch the graph of m against t. [2]

7 Given that $x^2 + xy + y^2 = 12$, find $\dfrac{dy}{dx}$ in terms of x and y. [5]

Section B (36 marks)

8 Fig. 8 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{1}{1 + \cos x}$, for $0 \leqslant x \leqslant \frac{1}{2}\pi$.

P is the point on the curve with x-coordinate $\frac{1}{3}\pi$.

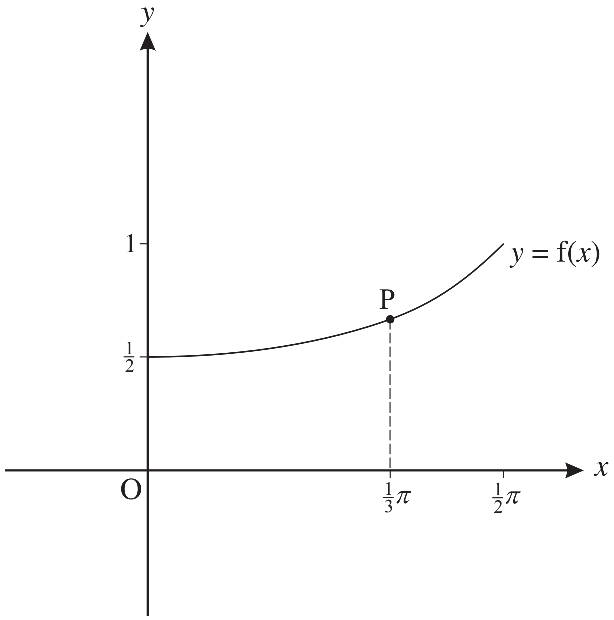


Fig. 8

(i) Find the y-coordinate of P. [1]

(ii) Find $\mathrm{f}'(x)$. Hence find the gradient of the curve at the point P. [5]

(iii) Show that the derivative of $\dfrac{\sin x}{1 + \cos x}$ is $\dfrac{1}{1 + \cos x}$. Hence find the exact area of the region enclosed by the curve $y = \mathrm{f}(x)$, the x-axis, the y-axis and the line $x = \frac{1}{3}\pi$. [7]

(iv) Show that $\mathrm{f}^{-1}(x) = \arccos\left(\dfrac{1}{x} - 1\right)$. State the domain of this inverse function, and add a sketch of $y = \mathrm{f}^{-1}(x)$ to a copy of Fig. 8. [5]

[Question 9 is printed overleaf.]

9 The function $f(x)$ is defined by $f(x) = \sqrt{4 - x^2}$ for $-2 \leqslant x \leqslant 2$.

(i) Show that the curve $y = \sqrt{4 - x^2}$ is a semicircle of radius 2, and explain why it is not the whole of this circle. [3]

Fig. 9 shows a point P (a, b) on the semicircle. The tangent at P is shown.

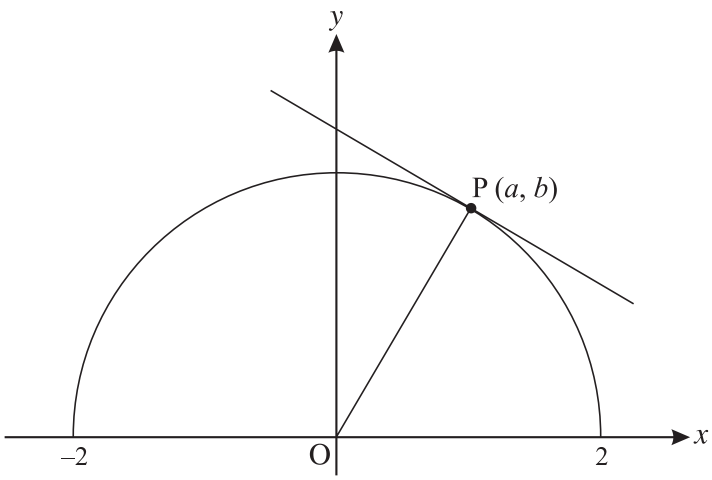


Fig. 9

(ii) (A) Use the gradient of OP to find the gradient of the tangent at P in terms of a and b.

(B) Differentiate $\sqrt{4 - x^2}$ and deduce the value of $f'(a)$.

(C) Show that your answers to parts (A) and (B) are equivalent. [6]

The function $g(x)$ is defined by $g(x) = 3f(x - 2)$, for $0 \leqslant x \leqslant 4$.

(iii) Describe a sequence of two transformations that would map the curve $y = f(x)$ onto the curve $y = g(x)$.

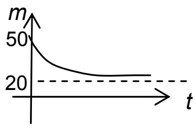
Hence sketch the curve $y = g(x)$. [6]

(iv) Show that if $y = g(x)$ then $9x^2 + y^2 = 36x$. [3]

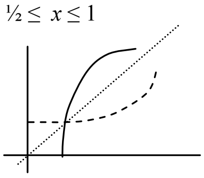
4753 (C3) Methods for Advanced Mathematics

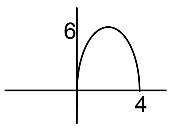
Section A

1 $\|2x-1\| \le 3$ $\Rightarrow$ $-3 \le 2x-1 \le 3$ $\Rightarrow$ $-2 \le 2x \le 4$ $\Rightarrow$ $-1 \le x \le 2$ *or* $(2x-1)^2 \le 9$ $\Rightarrow$ $4x^2-4x-8 \le 0$ $\Rightarrow$ $(4)(x+1)(x-2) \le 0$ $\Rightarrow$ $-1 \le x \le 2$	M1 A1 M1 A1 M1 A1 A1 A1 [4]	$2x-1 \le 3$ (or =) $x \le 2$ $2x-1 \ge -3$ (or =) $x \ge -1$ squaring and forming quadratic = 0 (or $\le$) factorising or solving to get $x = -1, 2$ $x \ge -1$ $x \le 2$ (www)
2 Let $u = x$, $dv/dx = e^{3x} \Rightarrow v = e^{3x}/3$ $\Rightarrow$ $\int x e^{3x}\,dx = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}.1.dx$ $= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$	M1 A1 A1 B1 [4]	parts with $u = x$, $dv/dx = e^{3x} \Rightarrow v$ $= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$ $+c$
3 (i) $f(-x) = f(x)$ Symmetrical about Oy.	B1 B1 [2]	
(ii) (A) even (B) neither (C) odd	B1 B1 B1 [3]	
4 Let $u = x^2 + 2 \Rightarrow du = 2x\,dx$ $\int_1^4 \frac{x}{x^2+2}dx = \int_3^{18} \frac{1/2}{u}du$ $= \frac{1}{2}[\ln u]_3^{18}$ $= \frac{1}{2}(\ln 18 - \ln 3)$ $= \frac{1}{2}\ln(18/3)$ $= \frac{1}{2}\ln 6$*	M1 A1 M1 E1 [4]	$\int \frac{1/2}{u}du$ or $k \ln(x^2+1)$ $\frac{1}{2}\ln u$ or $\frac{1}{2}\ln(x^2+2)$ substituting correct limits (u or x) must show working for ln 6
5 $y = x^2 \ln x$ $\Rightarrow$ $\frac{dy}{dx} = x^2.\frac{1}{x} + 2x\ln x$ $= x + 2x\ln x$ $dy/dx = 0$ when $x + 2x\ln x = 0$ $\Rightarrow$ $x(1 + 2\ln x) = 0$ $\Rightarrow$ $\ln x = -\frac{1}{2}$ $\Rightarrow$ $x = e^{-\frac{1}{2}} = 1/\sqrt{e}$ *	M1 B1 A1 M1 M1 E1 [6]	product rule $d/dx(\ln x) = 1/x$ soi oe their deriv = 0 or attempt to verify $\ln x = -\frac{1}{2} \Rightarrow x = e^{-\frac{1}{2}}$ or $\ln(1/\sqrt{e}) = -\frac{1}{2}$

6(i) Initial mass = 20 + 30 e^0 = 50 grams Long term mass = 20 grams	M1A1 B1 [3]	
(ii) $30 = 20 + 30\,e^{-0.1t}$ $\Rightarrow$ $e^{-0.1t} = 1/3$ $\Rightarrow$ $-0.1t = \ln(1/3) = -1.0986\ldots$ $\Rightarrow$ $t = 11.0$ mins	M1 M1 A1 [3]	 anti-logging correctly 11, 11.0, 10.99, 10.986 (not more than 3 d.p)
(iii) 	B1 B1 [2]	correct shape through (0, 50) – ignore negative values of t $\to 20$ as $t \to \infty$
7 $x^2 + xy + y^2 = 12$ $\Rightarrow$ $2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$ $\Rightarrow$ $(x+2y)\frac{dy}{dx} = -2x - y$ $\Rightarrow$ $\frac{dy}{dx} = -\frac{2x+y}{(x+2y)}$	M1 B1 A1 M1 A1 [5]	Implicit differentiation $x\frac{dy}{dx} + y$ correct equation collecting terms in dy/dx and factorising oe cao

Section B

8(i) $y = 1/(1+\cos\pi/3) = 2/3$.	B1 [1]	or 0.67 or better
(ii) $f'(x) = -1(1+\cos x)^{-2}.-\sin x$ $= \dfrac{\sin x}{(1+\cos x)^2}$ When $x = \pi/3$, $f'(x) = \dfrac{\sin(\pi/3)}{(1+\cos(\pi/3))^2}$ $= \dfrac{\sqrt{3}/2}{(1\frac{1}{2})^2} = \dfrac{\sqrt{3}}{2}\times\dfrac{4}{9} = \dfrac{2\sqrt{3}}{9}$	M1 B1 A1 M1 A1 [5]	chain rule or quotient rule d/dx (cos x) = –sin x soi correct expression substituting $x = \pi/3$ oe or 0.38 or better. (0.385, 0.3849)
(iii) deriv = $\dfrac{(1+\cos x)\cos x - \sin x.(-\sin x)}{(1+\cos x)^2}$ $= \dfrac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$ $= \dfrac{\cos x + 1}{(1+\cos x)^2}$ $= \dfrac{1}{1+\cos x}$ * Area $= \int_0^{\pi/3} \dfrac{1}{1+\cos x}dx$ $= \left[\dfrac{\sin x}{1+\cos x}\right]_0^{\pi/3}$ $= \dfrac{\sin\pi/3}{1+\cos\pi/3}(-0)$ $= \dfrac{\sqrt{3}}{2}\times\dfrac{2}{3} = \dfrac{\sqrt{3}}{3}$	M1 A1 M1dep E1 B1 M1 A1 cao [7]	Quotient or product rule – condone uv' – u'v for M1 correct expression $\cos^2 x + \sin^2 x = 1$ used dep M1 www substituting limits or $1/\sqrt{3}$ - must be exact
(iv) $y = 1/(1 + \cos x)$ $x \leftrightarrow y$ $x = 1/(1 + \cos y)$ $\Rightarrow$ $1 + \cos y = 1/x$ $\Rightarrow$ $\cos y = 1/x - 1$ $\Rightarrow$ $y = \arccos(1/x - 1)$ * Domain is $\frac{1}{2} \le x \le 1$ 	M1 A1 E1 B1 B1 [5]	attempt to invert equation www reasonable reflection in $y = x$

9 (i) $y = \sqrt{4 - x^2}$ $\Rightarrow$ $y^2 = 4 - x^2$ $\Rightarrow$ $x^2 + y^2 = 4$ which is equation of a circle centre O radius 2 Square root does not give negative values, so this is only a semi-circle.	M1 A1 B1 [3]	squaring $x^2 + y^2 = 4$ + comment (correct) oe, e.g. f is a function and therefore single valued
(ii) (*A*) Grad of OP = b/a $\Rightarrow$ grad of tangent = $-\frac{a}{b}$ (*B*) $f'(x) = \frac{1}{2}(4 - x^2)^{-1/2}.(-2x)$ $= -\frac{x}{\sqrt{4 - x^2}}$ $\Rightarrow$ $f'(a) = -\frac{a}{\sqrt{4 - a^2}}$ (*C*) $b = \sqrt{(4 - a^2)}$ so $f'(a) = -\frac{a}{b}$ as before	M1 A1 M1 A1 B1 E1 [6]	 chain rule or implicit differentiation oe substituting a into their $f'(x)$
(iii) Translation through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ followed by stretch scale factor 3 in y-direction 	M1 A1 M1 A1 M1 A1 [6]	Translation in x-direction through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ or 2 to right ('shift', 'move' M1 A0) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ alone is SC1 stretch in y-direction (condone y 'axis') (scale) factor 3 elliptical (or circular) shape through (0, 0) and (4, 0) and (2, 6) (soi) –1 if whole ellipse shown
(iv) $y = 3f(x - 2)$ $= 3\sqrt{(4 - (x - 2)^2)}$ $= 3\sqrt{(4 - x^2 + 4x - 4)}$ $= 3\sqrt{(4x - x^2)}$ $\Rightarrow$ $y^2 = 9(4x - x^2)$ $\Rightarrow$ $9x^2 + y^2 = 36x$ *	M1 A1 E1 [3]	or substituting $3\sqrt{(4 - (x - 2)^2)}$ oe for y in $9x^2 + y^2$ $4x - x^2$ www