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1)

$$|2x-1| \le 3$$

$$\Rightarrow \quad -3 \le 2x-1 \le 3$$

$$-3+1 \le 2x \le 3+1$$

$$-2 \le 2x \le 4$$

$$-\frac{2}{2} \le x \le \frac{4}{2}$$

$$-1 \le x \le 2$$

2)

$$\int x e^{3x}\,dx$$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

Let $u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{3x}$ $\Rightarrow v = \frac{1}{3}e^{3x}$

$$\int x e^{3x}dx = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}dx$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c$$

3) $f(x)$ is an even function

i) if $f(-x) = f(x)$ for all x in the domain

An even function is symmetrical about the y-axis

3ii) A) $f(x) = x^2 - 3$ even

B) $g(x) = \sin x + \cos x$ neither

C) $h(x) = \dfrac{1}{x+x^3}$ odd

4)

$$\int_1^4 \frac{x}{x^2+2}\,dx$$

$$= \frac{1}{2}\int_1^4 \frac{2x}{x^2+2}\,dx$$

$$= \frac{1}{2}\Big[\ln(x^2+2)\Big]_1^4$$

$$= \frac{1}{2}\Big[\ln 18 - \ln 3\Big]$$

$$= \frac{1}{2}\ln\left(\frac{18}{3}\right) = \frac{1}{2}\ln 6$$

This solution is based on the fact that when the numerator of a fraction is the differential of the denominator, then the integral of the fraction is the natural log of the denominator.

4) Alternative method using a formal substitution

$$\int_1^4 \frac{x}{x^2+2}\,dx$$

Let $u = x^2+2$

$$\frac{du}{dx} = 2x$$

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4 cont) $\Rightarrow \quad du = 2x\,dx$

$$\tfrac{1}{2}\,du = x\,dx$$

when $x = 1$, $u = 1^2+2 = 3$

when $x = 4$, $u = 4^2+2 = 18$

$$\therefore \int_1^4 \frac{x}{x^2+2}\,dx = \frac{1}{2}\int_3^{18} \frac{1}{u}\,du$$

$$= \frac{1}{2}\Big[\ln u\Big]_3^{18}$$

$$= \frac{1}{2}\Big[\ln 18 - \ln 3\Big]$$

$$= \frac{1}{2}\ln\left(\frac{18}{3}\right) = \frac{1}{2}\ln 6$$

5) $$y = x^2 \ln x$$

$$\frac{dy}{dx} = x^2\frac{1}{x} + 2x\ln x$$

$$\frac{dy}{dx} = x + 2x\ln x$$

$$= x(1 + 2\ln x)$$

when $x = \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$

$$\frac{dy}{dx} = e^{-\frac{1}{2}}\left(1 + 2\ln(e^{-\frac{1}{2}})\right)$$

$$= e^{-\frac{1}{2}}(1 - \ln e)$$

$$= e^{-\frac{1}{2}}(1 - 1) = 0$$

$\therefore$ y has a st. pt. at $x = \frac{1}{\sqrt{e}}$

6) $$m = 20 + 30e^{-0.1t}$$

i) Initial mass when $t = 0$

$$m = 20 + 30e^0$$

$$m = 50\text{ g}$$

As $t \to \infty$

$$m \to 20 + 30e^{-\infty}$$

$$m \to 20 + 0 = 20\text{g}$$

ii) $$30 = 20 + 30e^{-0.1t}$$

$$10 = 30e^{-0.1t}$$

$$\frac{10}{30} = e^{-0.1t}$$

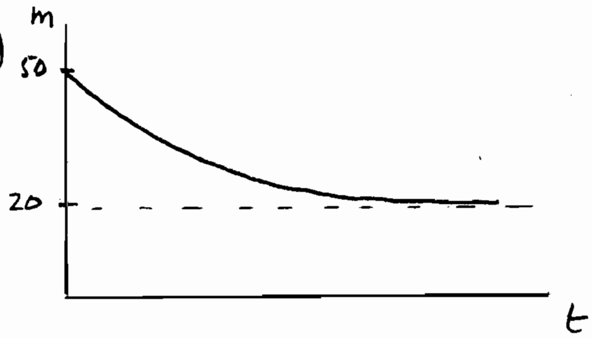
$$\ln\left(\frac{10}{30}\right) = -0.1t$$

$$\frac{\ln\left(\frac{1}{3}\right)}{-0.1} = t$$

$$t = 10.986$$

$t = 11.0$ minutes to 3 s.f

iii)



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7)

$$x^2 + xy + y^2 = 12$$

$$2x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$$

$$(x+2y)\frac{dy}{dx} = -y-2x$$

$$\frac{dy}{dx} = \frac{-y-2x}{x+2y}$$

or $-\dfrac{y+2x}{x+2y}$

8)

$$y = \frac{1}{1+\cos x} \qquad 0 \le x \le \frac{\pi}{2}$$

i) When $x = \frac{\pi}{3}$, $y = \dfrac{1}{1+\cos\frac{\pi}{3}}$

$$y = \frac{1}{1+\frac{1}{2}} = \frac{1}{\frac{3}{2}}$$

$$y = \frac{2}{3}$$

so $P\left(\frac{\pi}{3}, \frac{2}{3}\right)$

ii) $y = (1+\cos x)^{-1}$

$$\frac{dy}{dx} = -1(1+\cos x)^{-2}(-\sin x)$$

$$\frac{dy}{dx} = \frac{\sin x}{(1+\cos x)^2}$$

When $x = \frac{\pi}{3}$, $\dfrac{dy}{dx} = \dfrac{\sin\frac{\pi}{3}}{(1+\cos\frac{\pi}{3})^2}$

$$\frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2}}{(1+\frac{1}{2})^2} = \frac{\frac{\sqrt{3}}{2}}{\frac{9}{4}}$$

$$\frac{dy}{dx} = \frac{\sqrt{3}}{2} \times \frac{4}{9} = \frac{2\sqrt{3}}{9}$$

iii) $\dfrac{d}{dx}\dfrac{\sin x}{1+\cos x}$

$$\left(\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right)$$

$$= \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

$$\text{Area} = \int_0^{\frac{\pi}{3}} \frac{1}{1+\cos x}\,dx$$

$$= \left[\frac{\sin x}{1+\cos x}\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\sin\frac{\pi}{3}}{1+\cos\frac{\pi}{3}} - \frac{\sin 0}{1+\cos 0}$$

$$= \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} - 0$$

$$= \frac{\frac{\sqrt{3}}{2}}{3/2} = \frac{\sqrt{3}}{2} \times \frac{2}{3} = \frac{\sqrt{3}}{3}$$

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8 iv) $y = \dfrac{1}{1+\cos x}$

Swap variables and rearrange

$$x = \frac{1}{1+\cos y}$$

$$x(1+\cos y) = 1$$

$$1+\cos y = \frac{1}{x}$$

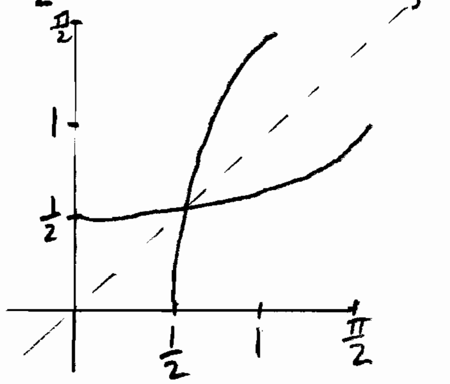
$$\cos y = \frac{1}{x} - 1$$

$$y = \cos^{-1}\left(\frac{1}{x} - 1\right)$$

$$\therefore f^{-1}(x) = \cos^{-1}\left(\frac{1}{x} - 1\right)$$

Domain of $f^{-1}(x)$ is given by the range of $f(x)$

$$\tfrac{1}{2} \le x \le 1$$



9) $f(x) = \sqrt{4-x^2} \quad -2 \le x \le 2$

$$y = \sqrt{4-x^2}$$

$$\Rightarrow y^2 = 4 - x^2$$

$$y^2 + x^2 = 2^2$$

This represents circle, centre $(0,0)$ radius 2.

However, $y = \sqrt{4-x^2}$ generates only values $y \ge 0$ so represents a semi-circle above the x-axis

ii) B) $y = (4-x^2)^{\frac{1}{2}}$

$$\frac{dy}{dx} = \tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{4-x^2}}$$

when $x = a$

$$\frac{dy}{dx} = -\frac{a}{\sqrt{4-a^2}}$$

A) Gradient of OP $= \dfrac{b-0}{a-0}$

$$= \frac{b}{a}$$

$\therefore$ gradient of tangent at P which is $\perp$ to radius OP

$$= -\frac{a}{b}$$

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c) The answers to A and B are equivalent because

$$\frac{dy}{dx} = \frac{-a}{\sqrt{4-a^2}}$$

but $y = \sqrt{4-x^2}$

so at $P(a,b)$ $b = \sqrt{4-a^2}$

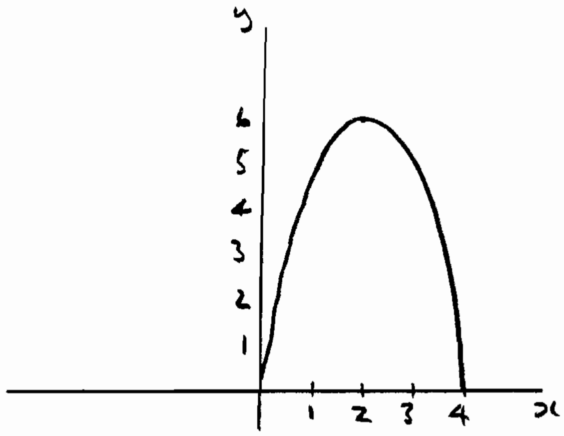
$$\therefore \frac{dy}{dx} = -\frac{a}{b}$$

which is same answer as A.

9 iii) $g(x) = 3f(x-2)$

$y=f(x)$ would be mapped to $y=g(x)$ by a translation by $\begin{pmatrix}2\\0\end{pmatrix}$ and a one-way stretch parallel to y-axis scale factor 3.

The transformations could be carried out in either order.



iv) $y = g(x)$

$\Rightarrow$ $y = 3f(x-2)$

$y = 3\sqrt{4-(x-2)^2}$

$y^2 = 9(4-(x-2)^2)$

$y^2 = 9(4-(x^2-4x+4))$

$y^2 = 9(4-x^2+4x-4)$

$y^2 = 9(4x-x^2)$

$y^2 = 36x - 9x^2$

$9x^2 + y^2 = 36x$