OCR
RECOGNISING ACHIEVEMENT

ADVANCED GCE
MATHEMATICS (MEI) 4753/01
Methods for Advanced Mathematics (C3)

Candidates answer on the Answer Booklet

OCR Supplied Materials:
- 8 page Answer Booklet
- MEI Examination Formulae and Tables (MF2)

Other Materials Required:
- Scientific or graphical calculator

Friday 11 June 2010
Morning

Duration: 1 hour 30 minutes

4753 01

INSTRUCTIONS TO CANDIDATES

- Write your name clearly in capital letters, your Centre Number and Candidate Number in the spaces provided on the Answer Booklet.
- Use black ink. Pencil may be used for graphs and diagrams only.
- Read each question carefully and make sure that you know what you have to do before starting your answer.
- Answer **all** the questions.
- Do **not** write in the bar codes.
- You are permitted to use a graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets **[ ]** at the end of each question or part question.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.
- The total number of marks for this paper is **72**.
- This document consists of **4** pages. Any blank pages are indicated.

Section A (36 marks)

1 Evaluate $\int_0^{\frac{1}{6}\pi} \cos 3x \, dx$. **[3]**

2 Given that $f(x) = |x|$ and $g(x) = x + 1$, sketch the graphs of the composite functions $y = fg(x)$ and $y = gf(x)$, indicating clearly which is which. **[4]**

3 **(i)** Differentiate $\sqrt{1 + 3x^2}$. **[3]**

(ii) Hence show that the derivative of $x\sqrt{1 + 3x^2}$ is $\dfrac{1 + 6x^2}{\sqrt{1 + 3x^2}}$. **[4]**

4 A piston can slide inside a tube which is closed at one end and encloses a quantity of gas (see Fig. 4).

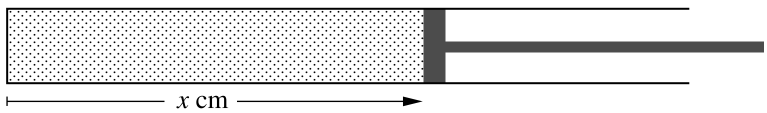


Fig. 4

The pressure of the gas in atmospheric units is given by $p = \dfrac{100}{x}$, where x cm is the distance of the piston from the closed end. At a certain moment, $x = 50$, and the piston is being pulled away from the closed end at 10 cm per minute. At what rate is the pressure changing at that time? **[6]**

5 Given that $y^3 = xy - x^2$, show that $\dfrac{dy}{dx} = \dfrac{y - 2x}{3y^2 - x}$.

Hence show that the curve $y^3 = xy - x^2$ has a stationary point when $x = \frac{1}{8}$. **[7]**

6 The function $f(x)$ is defined by

$$f(x) = 1 + 2\sin 3x, \quad -\frac{\pi}{6} \leqslant x \leqslant \frac{\pi}{6}.$$

You are given that this function has an inverse, $f^{-1}(x)$.

Find $f^{-1}(x)$ and its domain. **[6]**

7 State whether the following statements are true or false; if false, provide a counter-example.

(i) If a is rational and b is rational, then $a + b$ is rational.

(ii) If a is rational and b is irrational, then $a + b$ is irrational.

(iii) If a is irrational and b is irrational, then $a + b$ is irrational. **[3]**

Section B (36 marks)

8 Fig. 8 shows the curve $y = 3\ln x + x - x^2$.

The curve crosses the x-axis at P and Q, and has a turning point at R. The x-coordinate of Q is approximately 2.05.

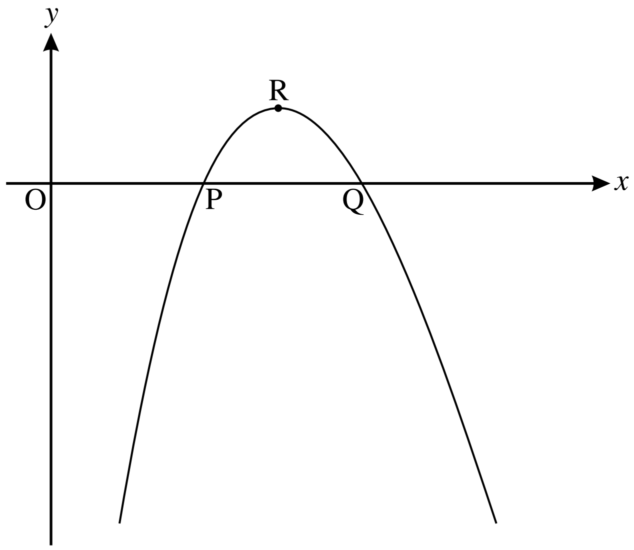


Fig. 8

(i) Verify that the coordinates of P are (1, 0). **[1]**

(ii) Find the coordinates of R, giving the y-coordinate correct to 3 significant figures.

Find $\frac{d^2y}{dx^2}$, and use this to verify that R is a maximum point. **[9]**

(iii) Find $\int \ln x \, dx$.

Hence calculate the area of the region enclosed by the curve and the x-axis between P and Q, giving your answer to 2 significant figures. **[7]**

[Question 9 is printed overleaf.]

9 Fig. 9 shows the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{\mathrm{e}^{2x}}{1 + \mathrm{e}^{2x}}$. The curve crosses the y-axis at P.

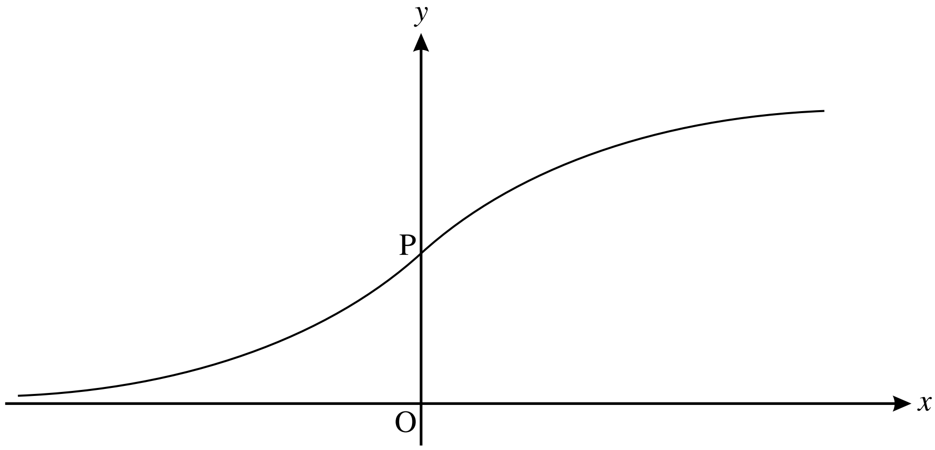


Fig. 9

(i) Find the coordinates of P. **[1]**

(ii) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, simplifying your answer.

Hence calculate the gradient of the curve at P. **[4]**

(iii) Show that the area of the region enclosed by $y = \mathrm{f}(x)$, the x-axis, the y-axis and the line $x = 1$ is $\frac{1}{2}\ln\left(\dfrac{1 + \mathrm{e}^2}{2}\right)$. **[5]**

The function $\mathrm{g}(x)$ is defined by $\mathrm{g}(x) = \frac{1}{2}\left(\dfrac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}}\right)$.

(iv) Prove algebraically that $\mathrm{g}(x)$ is an odd function.

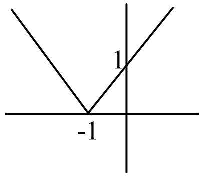
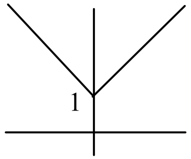
Interpret this result graphically. **[3]**

(v) (*A*) Show that $\mathrm{g}(x) + \frac{1}{2} = \mathrm{f}(x)$.

(*B*) Describe the transformation which maps the curve $y = \mathrm{g}(x)$ onto the curve $y = \mathrm{f}(x)$.

(*C*) What can you conclude about the symmetry of the curve $y = \mathrm{f}(x)$? **[6]**

Section A

1 $\int_0^{\pi/6} \cos 3x \, dx = \left[\frac{1}{3}\sin 3x\right]_0^{\pi/6}$	M1	$k \sin 3x$, $k > 0$, $k \neq 3$	or M1 for $u = 3x \Rightarrow \int \frac{1}{3}\cos u \, du$ condone 90° in limit
$= \frac{1}{3}\sin\frac{\pi}{2} - 0$	B1	$k = (\pm)1/3$	or M1 for $\left[\frac{1}{3}\sin u\right]$
$= 1/3$	A1cao [3]	0.33 or better	so: $\sin 3x$: M1B0, $-\sin 3x$: M0B0, $\pm 3\sin 3x$: M0B0, $-1/3 \sin 3x$: M0B1
2 $\text{fg}(x) = \lvert x+1 \rvert$ $\text{g f}(x) = \lvert x \rvert + 1$	B1 B1	soi from correctly-shaped graphs (i.e. without intercepts)	but must indicate which is which bod gf if negative x values are missing
 	B1 B1	graph of $\lvert x+1 \rvert$ only graph of $\lvert x \rvert + 1$	'V' shape with (−1, 0) and (0, 1) labelled 'V' shape with (0, 1) labelled (0, 1)
	[4]		
3(i) $y = (1+3x^2)^{1/2}$			
$\Rightarrow$ $dy/dx = \frac{1}{2}(1+3x^2)^{-1/2}.6x$	M1 B1	chain rule $\frac{1}{2} u^{-1/2}$	
$= 3x(1 + 3x^2)^{-1/2}$	A1 [3]	o.e., but must be '3'	can isw here
(ii) $y = x(1+3x^2)^{1/2}$			
$\Rightarrow$ $dy/dx = x.\frac{3x}{\sqrt{1+3x^2}} + 1.(1+3x^2)^{1/2}$	M1 A1ft	product rule ft their dy/dx from (i)	
$= \frac{3x^2 + 1 + 3x^2}{\sqrt{1+3x^2}}$	M1	common denominator or factoring $(1+3x^2)^{-1/2}$	must show this step for M1 E1
$= \frac{1+6x^2}{\sqrt{1+3x^2}}$ *	E1 [4]	www	

4 $p = 100/x = 100\,x^{-1}$ $\Rightarrow$ $dp/dx = -100x^{-2} = -100/x^2$ $dp/dt = dp/dx \times dx/dt$ $dx/dt = 10$ When $x = 50$, $dp/dx = (-100/50^2)$ $\Rightarrow dp/dt = 10 \times -0.04 = -0.4$	M1 A1 M1 B1 M1dep A1cao [6]	attempt to differentiate $-100x^{-2}$ o.e. o.e. soi soi substituting $x = 50$ into their dp/dx dep 2nd M1 o.e. e.g. decreasing at 0.4	condone poor notation if chain rule correct *or* $x = 50 + 10\,t$ B1 $\Rightarrow P = 100/x = 100/(50 + 10\,t)$ $\Rightarrow dP/dt = -100(50 + 10\,t)^{-2} \times 10 = -1000/(50 + 10t)^{-2}$ M1 A1 When $t = 0$, $dP/dt = -1000/50^2 = -0.4$ A1
5 $y^3 = xy - x^2$ $\Rightarrow$ $3y^2\,dy/dx = x\,dy/dx + y - 2x$ $\Rightarrow$ $3y^2\,dy/dx - x\,dy/dx = y - 2x$ $\Rightarrow$ $(3y^2 - x)\,dy/dx = y - 2x$ $\Rightarrow$ $dy/dx = (y - 2x)/(3y^2 - x)$ * TP when $dy/dx = 0 \Rightarrow y - 2x = 0$ $\Rightarrow$ $y = 2x$ $\Rightarrow$ $(2x)^3 = x.2x - x^2$ $\Rightarrow$ $8x^3 = x^2$ $\Rightarrow$ $x = 1/8$ *(or 0)	B1 B1 M1 E1 M1 M1 E1 [7]	$3y^2dy/dx$ $x\,dy/dx + y - 2x$ collecting terms in dy/dx only or $x = 1/8$ and $dy/dx = 0 \Rightarrow y = ¼$ or $(1/4)^3 = (1/8)(1/4) - (1/8)^2$ or verifying e.g. $1/64 = 1/64$	 must show '$x\,dy/dx + y$' on one side or $x = 1/8 \Rightarrow y^3 = (1/8)y - 1/64$ M1 verifying that $y = ¼$ is a solution (must show evidence*) M1 $\Rightarrow dy/dx = (¼ - 2(1/8))/(\ldots) = 0$ E1 *just stating that $y = ¼$ is M1 M0 E0
6 $f(x) = 1 + 2\sin 3x = y$ $x \leftrightarrow y$ $x = 1 + 2\sin 3y$ $\Rightarrow$ $\sin 3y = (x - 1)/2$ $\Rightarrow$ $3y = \arcsin[(x - 1)/2]$ $\Rightarrow y = \frac{1}{3}\arcsin\left[\frac{x-1}{2}\right]$ so $f^{-1}(x) = \frac{1}{3}\arcsin\left[\frac{x-1}{2}\right]$ Range of f is -1 to 3 $\Rightarrow$ $-1 \le x \le 3$	M1 A1 A1 A1 M1 A1 [6]	attempt to invert must be $y = \ldots$ or $f^{-1}(x) = \ldots$ or $-1 \le (x - 1)/2 \le 1$ must be 'x', not y or $f(x)$	at least one step attempted, or reasonable attempt at flow chart inversion (or any other variable provided same used on each side) condone <'s for M1 allow unsupported correct answers; -1 to 3 is M1 A0
7 (*A*) True , (*B*) True , (C) False Counterexample, e.g. $\sqrt{2} + (-\sqrt{2}) = 0$	B2,1,0 B1 [3]		

8(i) When $x = 1$, $y = 3 \ln 1 + 1 - 1^2$ $= 0$	E1 [1]		
(ii) $\frac{dy}{dx} = \frac{3}{x} + 1 - 2x$ $\Rightarrow$ At R, $\frac{dy}{dx} = 0 = \frac{3}{x} + 1 - 2x$ $\Rightarrow$ $3 + x - 2x^2 = 0$ $\Rightarrow$ $(3 - 2x)(1 + x) = 0$ $\Rightarrow$ $x = 1.5$, (or -1) $\Rightarrow$ $y = 3 \ln 1.5 + 1.5 - 1.5^2$ $= 0.466$ (3 s.f.) $\frac{d^2y}{dx^2} = -\frac{3}{x^2} - 2$ When $x = 1.5$, d^2y/dx^2 $(= -10/3)$ $< 0 \Rightarrow$ max	M1 A1cao M1 M1 A1 M1 A1cao B1ft E1 [9]	$d/dx (\ln x) = 1/x$ re-arranging into a quadratic = 0 factorising or formula or completing square substituting their x ft their dy/dx on equivalent work www – don't need to calculate 10/3	 SC1 for $x = 1.5$ unsupported, SC3 if verified but condone rounding errors on 0.466
(iii) Let $u = \ln x$, $du/dx = 1/x$ $dv/dx = 1$, $v = x$ $\Rightarrow$ $\int \ln x dx = x \ln x - \int x.\frac{1}{x} dx$ $= x \ln x - \int 1.dx$ $= x \ln x - x + c$ $\Rightarrow$ $A = \int_1^{2.05} (3 \ln x + x - x^2) dx$ $= \left[3x \ln x - 3x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_1^{2.05}$ $= -2.5057 + 2.833..$ $= 0.33$ (2 s.f.)	 M1 A1 A1 B1 B1ft M1dep A1 cao [7]	 parts condone no c correct integral and limits (soi) $\left[3 \times \text{their } 'x \ln x - x' + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]$ substituting correct limits dep 1st B1	 allow correct result to be quoted (SC3)

9(i) (0, ½)	B1 [1]	allow y = ½ , but not (x =) ½ or (½ ,0) nor P = 1/2	
(ii) $\frac{dy}{dx}=\frac{(1+e^{2x})2e^{2x}-e^{2x}.2e^{2x}}{(1+e^{2x})^2}$ $=\frac{2e^{2x}}{(1+e^{2x})^2}$ When x = 0, dy/dx = $2e^0/(1+e^0)^2$ = ½	M1 A1 A1 B1ft [4]	Quotient or product rule correct expression – condone missing bracket cao – mark final answer follow through their derivative	product rule: $\frac{dy}{dx}=e^{2x}.2e^{2x}(-1)(1+e^{2x})^{-2}+2e^{2x}(1+e^{2x})^{-1}$ $-\frac{2e^{2x}}{(1+e^{2x})^2}$ from $(u\mathrm{d}v-v\mathrm{d}u)/v^2$ SC1
(iii) $A=\int_0^1 \frac{e^{2x}}{1+e^{2x}}\,dx$ $=\left[\frac{1}{2}\ln(1+e^{2x})\right]_0^1$ *or* let $u = 1 + e^{2x}$, $du/dx = 2e^{2x}$ $\Rightarrow$ $A=\int_2^{1+e^2}\frac{1/2}{u}\,du=\left[\frac{1}{2}\ln u\right]_2^{1+e^2}$ $=\frac{1}{2}\ln(1+e^2)-\frac{1}{2}\ln 2$ $=\frac{1}{2}\ln\left[\frac{1+e^2}{2}\right]$ *	B1 M1 A1 M1 A1 M1 E1 [5]	correct integral and limits (soi) $k\ln(1+e^{2x})$ k = ½ or $v = e^{2x}$, $dv/dx = 2e^{2x}$ o.e. [½ ln u] or [½ ln (v + 1)] substituting correct limits www	condone no dx allow missing dx's or incompatible limits, but penalise missing brackets
(iv) $g(-x)=\frac{1}{2}\left[\frac{e^{-x}-e^{x}}{e^{-x}+e^{x}}\right]=-\frac{1}{2}\left[\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\right]=-g(x)$ Rotational symmetry of order 2 about O	M1 E1 B1 [3]	substituting $-x$ for x in g(x) completion www – taking out –ve must be clear must have 'rotational' 'about O', 'order 2' (oe)	not g($-x$) ≠ g(x). Condone use of f for g.
(v)(*A*) $g(x)+\frac{1}{2}=\frac{1}{2}.\frac{e^x-e^{-x}}{e^x+e^{-x}}+\frac{1}{2}=\frac{1}{2}.(\frac{e^x-e^{-x}+e^x+e^{-x}}{e^x+e^{-x}})$ $=\frac{1}{2}.(\frac{2e^x}{e^x+e^{-x}})$ $=\frac{e^x.e^x}{e^x(e^x+e^{-x})}=\frac{e^{2x}}{e^{2x}+1}=f(x)$ (*B*) Translation $\begin{pmatrix}0\\1/2\end{pmatrix}$ (*C*) Rotational symmetry [of order 2]about P	M1 A1 E1 M1 A1 B1 [6]	combining fractions (correctly) translation in y direction up ½ unit dep 'translation' used o.e. condone omission of 180°/order 2	allow 'shift', 'move' in correct direction for M1. $\begin{pmatrix}0\\1/2\end{pmatrix}$ alone is SC1.