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1) $\int_0^{\frac{\pi}{6}} \cos 3x \, dx$

$= \left[\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{3}\sin\frac{\pi}{2} - \frac{1}{3}\sin 0$

$= \frac{1}{3}\times 1 - \frac{1}{3}\times 0$

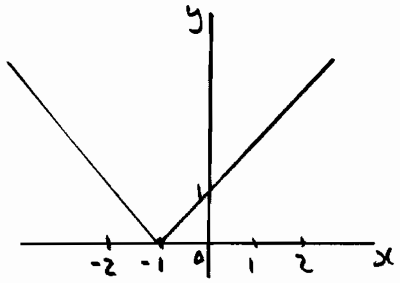
$= \frac{1}{3}$

2) $f(x) = |x|$, $g(x) = x+1$

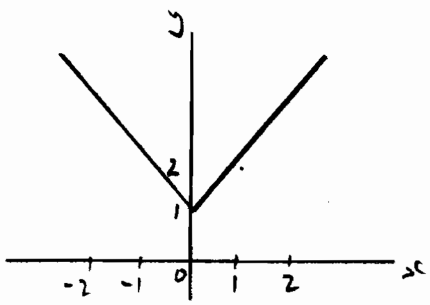
$fg(x) = f(x+1) = |x+1|$

$gf(x) = g(|x|) = |x|+1$

$y = fg(x) = |x+1|$




$y = gf(x) = |x|+1$




3) i) $\frac{d}{dx}\sqrt{1+3x^2}$

$= \frac{d}{dx}\left(1+3x^2\right)^{\frac{1}{2}}$

$= \frac{1}{2}\left(1+3x^2\right)^{-\frac{1}{2}} \times 6x$

$= \frac{3x}{\sqrt{1+3x^2}}$

3ii) $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d}{dx}\, x\sqrt{1+3x^2}$

$= x \times \frac{3x}{\sqrt{1+3x^2}} + \sqrt{1+3x^2} \times 1$

$= \frac{3x^2 + 1 + 3x^2}{\sqrt{1+3x^2}}$

$= \frac{6x^2+1}{\sqrt{1+3x^2}}$

4) $P = \frac{100}{x}$

When $x = 50$ cm

$\frac{dx}{dt} = 10$ cm per min

Find $\frac{dP}{dt}$

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4 cont)

$$\frac{dP}{dt} = \frac{dP}{dx} \times \frac{dx}{dt}$$

Now $P = \frac{100}{x} = 100x^{-1}$

so $\frac{dP}{dx} = -100x^{-2}$

$$\frac{dP}{dx} = -\frac{100}{x^2}$$

$$\therefore \frac{dP}{dt} = -\frac{100}{x^2} \times \frac{dx}{dt}$$

When $x = 50$ cm

$$\frac{dP}{dt} = -\frac{100}{50^2} \times 10$$

$\frac{dP}{dt} = -0.4$ atmospheric units per min

5) $y^3 = xy - x^2$

$$3y^2\frac{dy}{dx} = x\frac{dy}{dx} + y \times 1 - 2x$$

$$3y^2\frac{dy}{dx} - x\frac{dy}{dx} = y - 2x$$

$$(3y^2 - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{3y^2 - x}$$

At st. pt. $\frac{dy}{dx} = 0$

$$\Rightarrow y - 2x = 0$$

$$\Rightarrow y = 2x$$

For st.pt when $x = \frac{1}{8}$

$$y = 2 \times \frac{1}{8} = \frac{1}{4}$$

The point $\left(\frac{1}{8}, \frac{1}{4}\right)$ would need to be on curve

$$\left(\frac{1}{4}\right)^3 = \frac{1}{8} \times \frac{1}{4} - \left(\frac{1}{8}\right)^2$$

$$\frac{1}{64} = \frac{1}{32} - \frac{1}{64} \quad \checkmark$$

$\therefore \left(\frac{1}{8}, \frac{1}{4}\right)$ is on curve and $\frac{dy}{dx} = 0$ when $x = \frac{1}{8}$

$\therefore$ st pt when $x = \frac{1}{8}$

Alternative solution:

At st.pt $\frac{dy}{dx} = 0$

$$\Rightarrow y - 2x = 0$$

$$\Rightarrow y = 2x$$

Subst for $y$ in

$$y^3 = xy - x^2$$

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5 cont)

$$(2x)^3 = x(2x) - x^2$$
$$8x^3 = 2x^2 - x^2$$
$$8x^3 = x^2$$
$$8x^3 - x^2 = 0$$
$$x^2(8 - x) = 0$$
$$\Rightarrow \quad x = 0 \quad \text{or} \quad 8 - x = 0$$
$$x = \frac{1}{8}$$

$\therefore$ st. pt. when $x = \frac{1}{8}$

6) $f(x) = 1 + 2\sin 3x$

$-\frac{\pi}{6} \leq x \leq \frac{\pi}{6}$

Let $y = 1 + 2\sin 3x$

Swap variables and rearrange

$$x = 1 + 2\sin 3y$$
$$x - 1 = 2\sin 3y$$
$$\frac{x-1}{2} = \sin 3y$$
$$\sin^{-1}\left(\frac{x-1}{2}\right) = 3y$$
$$y = \frac{1}{3}\sin^{-1}\left(\frac{x-1}{2}\right)$$

so $f^{-1}(x) = \frac{1}{3}\sin^{-1}\left(\frac{x-1}{2}\right)$

Domain of $f^{-1}(x)$ is the range of $f(x)$

when $x = -\frac{\pi}{6}$, $f(x) = 1 + 2\sin(-\frac{\pi}{2})$
$= 1 - 2 = -1$

when $x = \frac{\pi}{6}$, $f(x) = 1 + 2\sin\frac{\pi}{2}$
$= 1 + 2 = 3$

Range of $f(x)$

$$-1 \leq f(x) \leq 3$$

so domain of $f^{-1}(x)$ is given by:

$$-1 \leq x \leq 3$$

7) i) a is rational and b is rational

$\Rightarrow$ a+b is rational

TRUE.

ii) a rational and b irrational

$\Rightarrow$ a+b is irrational

TRUE

iii) a irrational, b irrational

$\Rightarrow$ a+b irrational

FALSE

Counter-example

$a = 2 + \sqrt{3}$, $b = 2 - \sqrt{3}$
are both irrational

But $a + b = 4$ which is rational

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## Section B

8) $y = 3\ln x + x - x^2$

i) When $x = 1$

$y = 3\ln 1 + 1 - 1^2$

$= 0 + 1 - 1 = 0$

$\therefore$ P is $(1, 0)$

ii)

$\frac{dy}{dx} = \frac{3}{x} + 1 - 2x$

At st. pt. R $\frac{dy}{dx} = 0$

$\Rightarrow \frac{3}{x} + 1 - 2x = 0$

$3 + x - 2x^2 = 0$

$2x^2 - x - 3 = 0$

$(2x - 3)(x + 1) = 0$

$\Rightarrow x = \frac{3}{2}$ or $x = -1$

y undefined for $x = -1$

$\Rightarrow x = \frac{3}{2}$

$y = 3\ln\frac{3}{2} + \frac{3}{2} - \left(\frac{3}{2}\right)^2$

$= 0.466395$

$= 0.466$ to 3 s.f.

$R\left(\frac{3}{2}, 0.466\right)$

$\frac{dy}{dx} = \frac{3}{x} + 1 - 2x$

$\frac{dy}{dx} = 3x^{-1} + 1 - 2x$

$\frac{d^2y}{dx^2} = -3x^{-2} - 2$

$\frac{d^2y}{dx^2} = -\frac{3}{x^2} - 2$

When $x = \frac{3}{2}$

$\frac{d^2y}{dx^2} = -\frac{3}{\frac{9}{4}} - 2$

$= -\frac{4}{3} - 2 = -\frac{10}{3} < 0$

Since $\frac{d^2y}{dx^2} < 0$

st. pt. at R is a maximum

iii) $\int \ln x \, dx$

Let $u = \ln x$ Let $\frac{dv}{dx} = 1$

$\Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\Rightarrow v = x$

Using $\int u \frac{dv}{dx} = uv - \int v \frac{du}{dx}$

$\int \ln x \, dx = x\ln x - \int x \cdot \frac{1}{x} \, dx$

$= x\ln x - \int 1 \, dx$

$= x\ln x - x + c$

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8 iii cont) Area $= \int_1^{2.05} (3\ln x + x - x^2)\,dx$

$= \left[3(x\ln x - x) + \frac{x^2}{2} - \frac{x^3}{3}\right]_{1.5}^{2.05}$

$= \left(3(2.05\ln 2.05 - 2.05) + \frac{2.05^2}{2} - \frac{2.05^3}{3}\right)$

$- \left(3(1\ln 1 - 1) + \frac{1^2}{2} - \frac{1^3}{3}\right)$

$= (-2.5057) - (-2.8333)$

$= 0.3276$

$= 0.33$ to 2 s.f.

9) i) $y = f(x) = \dfrac{e^{2x}}{1+e^{2x}}$

when $x = 0$,

$y = \dfrac{e^0}{1+e^0} = \dfrac{1}{1+1} = \dfrac{1}{2}$

$\therefore$ P is $(0, \frac{1}{2})$

ii) $\dfrac{d}{dx}\dfrac{u}{v} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\dfrac{dy}{dx} = \dfrac{(1+e^{2x}) \times 2e^{2x} - e^{2x}(2e^{2x})}{(1+e^{2x})^2}$

$\dfrac{dy}{dx} = \dfrac{2e^{2x} + 2e^{4x} - 2e^{4x}}{(1+e^{2x})^2}$

$\dfrac{dy}{dx} = \dfrac{2e^{2x}}{(1+e^{2x})^2}$

When $x = 0$

$\dfrac{dy}{dx} = \dfrac{2 \times 1}{(1+1)^2} = \dfrac{1}{2}$

Gradient at P $= \frac{1}{2}$

iii) Area $= \int_0^1 \dfrac{e^{2x}}{1+e^{2x}}\,dx$

$= \frac{1}{2}\int_0^1 \dfrac{2e^{2x}}{1+e^{2x}}\,dx$

(numerator is now differential of denominator so integral is log of denominator)

$= \frac{1}{2}\left[\ln(1+e^{2x})\right]_0^1$

$= \frac{1}{2}\left[\ln(1+e^2) - \ln(1+1)\right]$

$= \frac{1}{2}\ln\left(\dfrac{1+e^2}{2}\right)$

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9 iv) $g(x) = \frac{1}{2}\left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right)$

$g(-x) = \frac{1}{2}\left(\frac{e^{-x} - e^{x}}{e^{-x} + e^{x}}\right)$

$= \frac{1}{2}\left(\frac{-(e^{x} - e^{-x})}{e^{x} + e^{-x}}\right)$

$= -\frac{1}{2}\left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right)$

$= -g(x)$ for all $x \in \mathbb{R}$

$\therefore$ $g(x)$ is an odd function

Graphically, $g(x)$ has rotational symmetry of order 2 about the origin

9 v) A)

$g(x) + \frac{1}{2}$

$= \frac{1}{2}\left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}\right) + \frac{1}{2}$

$= \frac{1}{2}\left(\frac{e^{2x} - 1}{e^{2x} + 1}\right) + \frac{1}{2}$

(multiplying $g(x)$ by $\frac{e^{x}}{e^{x}}$)

$= \frac{1}{2}\left[\frac{e^{2x} - 1}{e^{2x} + 1} + \frac{e^{2x} + 1}{e^{2x} + 1}\right]$

$= \frac{1}{2}\left[\frac{2e^{2x}}{e^{2x} + 1}\right] = \frac{e^{2x}}{e^{2x} + 1} = f(x)$

B)

$g(x) + \frac{1}{2} = f(x)$

Translation by $\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$ will map $g(x)$ onto $f(x)$

C)

$f(x)$ will have rotational symmetry of order 2 about the point $(0, \frac{1}{2})$