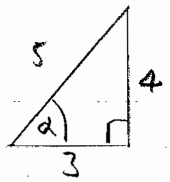
1) $3\cos\theta + 4\sin\theta$

$= 5\left(\frac{3}{5}\cos\theta + \frac{4}{5}\sin\theta\right)$



$= 5\left(\cos\alpha\cos\theta + \sin\alpha\sin\theta\right)$

$= 5\cos(\theta - \alpha)$

where $\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.1°$

$= 5\cos(\theta - 53.1°)$

$f(\theta) = 7 + 5\cos(\theta - 53.1°)$

Range $2 \le f(\theta) \le 12$

Max value of $\dfrac{1}{7 + 3\cos\theta + 4\sin\theta}$

$= \dfrac{1}{7 + (-5)} = \dfrac{1}{2}$

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2) $\sqrt{4+2x} = (4+2x)^{\frac{1}{2}}$

$= \left(4\left(1 + \frac{x}{2}\right)\right)^{\frac{1}{2}}$

$= 4^{\frac{1}{2}}\left(1 + \frac{x}{2}\right)^{\frac{1}{2}}$

$= 2\left[1 + \frac{1}{2}\left(\frac{x}{2}\right) + \frac{\frac{1}{2}\cdot -\frac{1}{2}}{1\cdot 2}\left(\frac{x}{2}\right)^2 + \frac{\frac{1}{2}\cdot -\frac{1}{2}\cdot -\frac{3}{2}}{1\cdot 2\cdot 3}\left(\frac{x}{2}\right)^3\right]$

$= 2\left[1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} - + \right]$

$= 2 + \frac{x}{2} - \frac{x^2}{16} + \frac{x^3}{64}$

to first 4 terms

Expansion valid for

$\left|\frac{x}{2}\right| < 1$

$|x| < 2$

$-2 < x < 2$

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3) $\sec^2\theta = 4$

$1 + \tan^2\theta = 4$

$\tan^2\theta = 3$

$\tan\theta = \pm\sqrt{3}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ for $0 < \theta < \pi$

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4) $y = \sqrt{1 + e^{-2x}}$

$V = \pi\int_0^1 y^2\,dx$

$V = \pi\int_0^1 (1 + e^{-2x})\,dx$

$= \pi\left[x - \frac{1}{2}e^{-2x}\right]_0^1$

$= \pi\left[\left(1 - \frac{1}{2e^2}\right) - \left(0 - \frac{1}{2}\right)\right]$

$= \pi\left[\frac{3}{2} - \frac{1}{2e^2}\right]$

5) $2\cos 2x = 1 + \cos x$

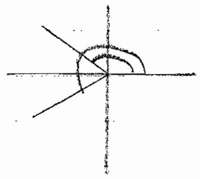
$2(2\cos^2 x - 1) = 1 + \cos x$

$4\cos^2 x - 2 - \cos x - 1 = 0$

$4\cos^2 x - \cos x - 3 = 0$

$(4\cos x + 3)(\cos x - 1) = 0$

$\Rightarrow \cos x = -\frac{3}{4}$ or $\cos x = 1$



$x = 138.6^\circ$
$x = 221.4^\circ$
$x = 0^\circ$

for $0 \le x < 360^\circ$

6) i) $y^2 - x^2 = 4$

Verify $x = t - \frac{1}{t}$, $y = t + \frac{1}{t}$

are parametric eqns of curve

Substituting

$\left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2$

$= t^2 + 2 + \frac{1}{t^2} - \left(t^2 - 2 + \frac{1}{t^2}\right)$

$= t^2 + 2 + \frac{1}{t^2} - t^2 + 2 - \frac{1}{t^2}$

$= 4$ as required

6ii) $\frac{dy}{dt} = 1 - \frac{1}{t^2}$, $\frac{dx}{dt} = 1 + \frac{1}{t^2}$

$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$

$= \dfrac{1 - \frac{1}{t^2}}{1 + \frac{1}{t^2}}$

$= \dfrac{\frac{t^2 - 1}{t^2}}{\frac{t^2 + 1}{t^2}}$

$= \dfrac{t^2 - 1}{t^2 + 1}$

$= \dfrac{(t-1)(t+1)}{t^2 + 1}$

At st pt $\frac{dy}{dx} = 0$

$\Rightarrow (t-1)(t+1) = 0$

$\Rightarrow t = 1$ or $t = -1$

when $t = 1$, $y = 1 + \frac{1}{1} = 2$

$x = 1 - \frac{1}{1} = 0$

$\therefore (0, 2)$ a stationary point

when $t = -1$, $y = -1 + \frac{1}{-1} = -2$

$x = -1 - \frac{1}{-1} = 0$

$\therefore (0, -2)$ a stationary point

Section B

7) i)

$$\int \frac{t}{1+t^2}\,dt$$

$$= \frac{1}{2}\ln(1+t^2) + c$$

ii)

$$\frac{1}{t(1+t^2)} \equiv \frac{A}{t} + \frac{Bt+C}{1+t^2}$$

$$\Rightarrow 1 \equiv A(1+t^2) + (Bt+C)t$$

When $t = 0$

$$1 = A(1+0) \Rightarrow A = 1$$

Equating coeffs of t^2

$$0 = 1 + B \Rightarrow B = -1$$

Equating coeffs of t

$$0 = C \Rightarrow C = 0$$

$$\frac{1}{t(1+t^2)} \equiv \frac{1}{t} - \frac{t}{1+t^2}$$

iii)

$$\frac{dM}{dt} = \frac{M}{t(1+t^2)}$$

$$\int \frac{1}{M}\,dM = \int \frac{1}{t(1+t^2)}\,dt$$

$$\int \frac{1}{M}\,dM = \int\left(\frac{1}{t} - \frac{t}{1+t^2}\right)dt$$

$$\ln M = \ln t - \frac{1}{2}\ln(1+t^2) + c$$

$$\ln M = \ln t - \ln(1+t^2)^{\frac{1}{2}} + \ln k$$

where $\ln k$ is a constant

$$\ln M = \ln\left(\frac{kt}{(1+t^2)^{\frac{1}{2}}}\right)$$

$$\Rightarrow M = \frac{kt}{\sqrt{1+t^2}}$$

iv) When $t = 1$, $M = 25$

$$25 = \frac{k}{\sqrt{2}}$$

$$\Rightarrow k = 25\sqrt{2}$$

$$\therefore M = \frac{25\sqrt{2}\,t}{\sqrt{1+t^2}}$$

As $t \to \infty$

$$\sqrt{1+t^2} \to \sqrt{t^2} = t$$

$\therefore M \to 25\sqrt{2}$ grams

8)

i) $P(0, 10, 30)$

$Q(0, 20, 15)$

$R(-15, 20, 30)$

$$\vec{PQ} = \begin{pmatrix} 0-0 \\ 20-10 \\ 15-30 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ -15 \end{pmatrix}$$

$$\vec{PR} = \begin{pmatrix} -15-0 \\ 20-10 \\ 30-30 \end{pmatrix} = \begin{pmatrix} -15 \\ 10 \\ 0 \end{pmatrix}$$

8 ii) $\begin{pmatrix}2\\3\\2\end{pmatrix}.\overrightarrow{PQ} = \begin{pmatrix}2\\3\\2\end{pmatrix}.\begin{pmatrix}0\\10\\-15\end{pmatrix}$

$= 0 + 30 - 30 = 0$

$\therefore \begin{pmatrix}2\\3\\2\end{pmatrix}$ is $\perp$ to $\overrightarrow{PQ}$

$\begin{pmatrix}2\\3\\2\end{pmatrix}.\overrightarrow{PR} = \begin{pmatrix}2\\3\\2\end{pmatrix}.\begin{pmatrix}-15\\10\\0\end{pmatrix}$

$= -30 + 30 + 0 = 0$

$\therefore \begin{pmatrix}2\\3\\2\end{pmatrix}$ is $\perp$ to $\overrightarrow{PR}$

Since $\begin{pmatrix}2\\3\\2\end{pmatrix}$ is $\perp$ to 2 non-parallel lines in the plane PQR, it is a normal to the plane.

Plane is of form

$2x + 3y + 2z = d$

$P(0, 10, 30)$ on plane so

$2\times 0 + 3\times 10 + 2\times 30 = d$

$\Rightarrow d = 90$

Plane PQR is given by

$2x + 3y + 2z = 90$

iii) $S\left(\frac{0-15}{2}, \frac{20+20}{2}, \frac{15+30}{2}\right)$

$S\left(-\frac{15}{2}, 20, \frac{45}{2}\right)$

$\overrightarrow{PS} = \begin{pmatrix}-\frac{15}{2}\\10\\-\frac{15}{2}\end{pmatrix}$

$\overrightarrow{OT} = \overrightarrow{OP} + \frac{2}{3}\overrightarrow{PS}$

$\overrightarrow{OT} = \begin{pmatrix}0\\10\\30\end{pmatrix} + \frac{2}{3}\begin{pmatrix}-\frac{15}{2}\\10\\-\frac{15}{2}\end{pmatrix}$

$\overrightarrow{OT} = \begin{pmatrix}-5\\16\frac{2}{3}\\25\end{pmatrix}$

$\therefore$ T is point $(-5, 16\frac{2}{3}, 25)$

iv) Line of drill hole

$\underline{r} = \overrightarrow{OT} + \lambda\begin{pmatrix}2\\3\\2\end{pmatrix}$

$\underline{r} = \begin{pmatrix}-5\\50/3\\25\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\2\end{pmatrix}$

$C(-30, 0, 0)$

If C on drill hole line

$$\left.\begin{aligned} -5 + 2\lambda &= -30 \\ 50/3 + 3\lambda &= 0 \\ 25 + 2\lambda &= 0 \end{aligned}\right\} \quad \begin{matrix}①\\②\\③\end{matrix}$$

From ① $\lambda = -\frac{25}{2}$

From ② $\lambda = -\frac{50}{9}$

From ③ $\lambda = -\frac{25}{2}$

No common value of λ $\therefore$ line does not pass thro C