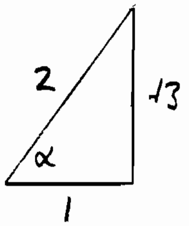
1)

$y = \sin x - \sqrt{3}\cos x$



$y = 2\left(\frac{1}{2}\sin x - \frac{\sqrt{3}}{2}\cos x\right)$

$y = 2\sin(x - \alpha)$

where $\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$\alpha = \frac{\pi}{3}$

P is point where y has max value and so angle is $\frac{\pi}{2}$

$x - \frac{\pi}{3} = \frac{\pi}{2}$

$x = \frac{5\pi}{6}$

P is point $\left(\frac{5\pi}{6}, 2\right)$

2)

i) $\frac{3 + 2x^2}{(1+x)^2(1-4x)} \equiv \frac{A}{1+x} + \frac{B}{(1+x)^2} + \frac{C}{1-4x}$

$\Rightarrow 3 + 2x^2 = A(1+x)(1-4x) + B(1-4x) + C(1+x)^2$

When $x = -1$

$3 + 2(-1)^2 = B(1 - 4(-1))$

$3 + 2 = 5B$

$\Rightarrow B = 1$

When $x = \frac{1}{4}$

$3 + 2\left(\frac{1}{4}\right)^2 = C\left(\frac{5}{4}\right)^2$

$3 + \frac{2}{16} = \frac{25C}{16}$

$\frac{50}{16} = \frac{25C}{16}$

$\Rightarrow C = 2$

When $x = 0$

$3 = A(1)(1) + 1(1) + 2(1)^2$

$3 = A + 1 + 2 \qquad \Rightarrow A = 0$

$\therefore A = 0, \quad B = 1, \quad C = 2$

2ii) $(1+x)^{-2} = 1 + -2x + \frac{-2.-3}{1.2}x^2 + \ldots$

$\approx 1 - 2x + 3x^2 \ldots$

$(1-4x)^{-1} = 1 + -1(-4x) + \frac{-1.-2}{1.2}(-4x)^2 + \ldots$

$\approx 1 + 4x + 16x^2 \ldots$

$\frac{3 + 2x^2}{(1+x)^2(1-4x)} = \frac{1}{(1+x)^2} + \frac{2}{1-4x}$

$= (1+x)^{-2} + 2(1-4x)^{-1}$

$\approx 1 - 2x + 3x^2 + 2(1 + 4x + 16x^2)$

$= 3 + 6x + 35x^2$

3)

$\sin(\theta + \alpha) = 2\sin\theta$

$\Rightarrow \sin\theta\cos\alpha + \cos\theta\sin\alpha = 2\sin\theta$

Divide by $\cos\theta$

$\tan\theta\cos\alpha + \sin\alpha = 2\tan\theta$

3 cont $\Rightarrow$

$$\sin\alpha = 2\tan\theta - \tan\theta\cos\alpha$$

$$\sin\alpha = \tan\theta(2-\cos\alpha)$$

$$\frac{\sin\alpha}{2-\cos\alpha} = \tan\theta$$

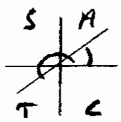
Solve $\sin(\theta + 40^\circ) = 2\sin\theta$

From above

$$\tan\theta = \frac{\sin 40^\circ}{2-\cos 40^\circ}$$

$$\tan\theta = 0.5209$$

$$\theta = 27.5^\circ$$



Also $\theta = 207.5^\circ$

4) a)

$$\frac{dx}{dt} = k\sqrt{x}$$

b)

$$\frac{dy}{dt} = \frac{10000}{\sqrt{y}}$$

$$\int \sqrt{y}\,dy = \int 10000\,dt$$

$$\int y^{\frac{1}{2}}\,dy = \int 10000\,dt$$

$$\frac{y^{3/2}}{\frac{3}{2}} = 10000t + c$$

$$\frac{2}{3}y^{3/2} = 10000t + c$$

Given $y = 900$ when $t = 0$

$$\frac{2}{3}900^{3/2} = 10000 \times 0 + c$$

$$18000 = c$$

$$\therefore \frac{2}{3}y^{3/2} = 10000t + 18000$$

$$y^{3/2} = 15000t + 27000$$

$$y = (15000t + 27000)^{2/3}$$

When $t = 10$

$$y = (15000 \times 10 + 27000)^{2/3}$$

$$y = 3152.45$$

$$y = 3152$$

5) Find $\int xe^{-2x}\,dx$

Let

$u = x$ $\Rightarrow \frac{du}{dx} = 1$

Let $\frac{dv}{dx} = e^{-2x}$ $\Rightarrow v = -\frac{1}{2}e^{-2x}$

Using $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

$$\int xe^{-2x}\,dx = -\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$$

$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c$$

$$= -\frac{1}{4}e^{-2x}(1+2x) + c$$

5ii) $\text{Vol} = \pi \int y^2 \, dx$

$$= \pi \int_0^2 \left(x^{1/2} e^{-x}\right)^2 dx$$

$$= \pi \int_0^2 x e^{-2x} \, dx$$

$$= \pi \left[-\frac{1}{4} e^{-2x} (1+2x) \right]_0^2$$

$$= \pi \left[\left(-\frac{1}{4} e^{-4} (1+4)\right) - \left(-\frac{1}{4} e^{0} (1)\right) \right]$$

$$= \frac{\pi}{4} \left[-\frac{5}{e^4} + 1 \right]$$

$$= \frac{\pi}{4} \left(1 - \frac{5}{e^4} \right)$$

6) $x = a(\theta - \sin\theta), \; y = a(1 - \cos\theta)$

i)

A) Find OE

At E x has max value

$y = 0$

$y = 0 \Rightarrow 1 - \cos\theta = 0$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0 \text{ or } 2\pi$

O is point $(0, 0)$

E is point $\left(a(2\pi - \sin 2\pi), 0\right)$

E is $(2a\pi, 0)$

$\therefore \; OE = 2a\pi - 0 = 2a\pi$

B) Max $y = a(1 - \cos\theta)$

when $\cos\theta = -1$

Max height $= a(1 - -1) = 2a$

ii)

$$\frac{dy}{d\theta} = +a\sin\theta$$

$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta}$$

$$= \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

iii) At B, AB is a tgt

Grad of AB $= \tan^{-1} 30^\circ = \frac{1}{\sqrt{3}}$

$\therefore$ gradient of curve at B $= \frac{1}{\sqrt{3}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{\sin\theta}{1-\cos\theta} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{3}}(1 - \cos\theta)$$

6 iii) cont)

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\frac{1}{\sqrt{3}}\left(1-\cos\frac{2\pi}{3}\right) = \frac{1}{\sqrt{3}}\left(1 - -\tfrac{1}{2}\right)$$

$$= \frac{1}{\sqrt{3}} \times \frac{3}{2} = \frac{\sqrt{3}}{2}$$

$\therefore$ $\theta = \frac{2\pi}{3}$ is a solution of

$$\sin\theta = \frac{1}{\sqrt{3}}(1-\cos\theta)$$

BF is y coord of B

$$y = a(1-\cos\theta)$$

at B, $\theta = \frac{2\pi}{3}$ so

$$y = a\left(1-\cos\frac{2\pi}{3}\right)$$

$$= a\left(1 - -\tfrac{1}{2}\right) = \frac{3a}{2}$$

$\Rightarrow$ $BF = \frac{3a}{2}$

OF is same as x coord of B

$$x = a(\theta - \sin\theta)$$

$$x = a\left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$$

$$x = a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$\therefore$ $OF = a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

6iv) Due to symmetry

$$BC = OE - 2 \times OF$$

$$= 2a\pi - 2\left(a\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)\right)$$

$$= 2a\pi - \frac{4a\pi}{3} + \sqrt{3}a$$

$$= \frac{2a\pi}{3} + \sqrt{3}a$$

$$BC = a\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{BF}{AF}$$

$\Rightarrow$ $AF = \sqrt{3}\,BF$

$$AF = \sqrt{3} \times \frac{3a}{2}$$

$$AF = \frac{3\sqrt{3}a}{2}$$

Due to symmetry

$$AD = 2AF + BC$$

$$= 2\left(\frac{3\sqrt{3}a}{2}\right) + a\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

$\therefore$ $20 = 3\sqrt{3}a + a\left(\frac{2\pi}{3} + \sqrt{3}\right)$

$$20 = a\left(4\sqrt{3} + \frac{2\pi}{3}\right)$$

$a = \dfrac{20}{\left(4\sqrt{3} + \frac{2\pi}{3}\right)} = 2.22\text{ m}$ to 3 s.f.

7i) $A(0,0,6)$ $E(15,-20,6)$

$|AE| = \sqrt{15^2+(-20)^2+(6-6)^2}$

$|AE| = 25\text{ m}$

7ii) Direction BD same as direction AE

$$\vec{AE} = \begin{pmatrix}15\\-20\\0\end{pmatrix} = 5\begin{pmatrix}3\\-4\\0\end{pmatrix}$$

Vector eqn of BD

$$\underline{r} = \begin{pmatrix}-1\\-7\\11\end{pmatrix} + \lambda\begin{pmatrix}3\\-4\\0\end{pmatrix}$$

$$|BD| = 15 \Rightarrow \left|\lambda\begin{pmatrix}3\\-4\\0\end{pmatrix}\right| = 15$$

$$\lambda\sqrt{3^2+(-4)^2+0^2} = 15$$

$$5\lambda = 15$$

$$\lambda = 3$$

D is given by

$$\underline{r} = \begin{pmatrix}-1\\-7\\11\end{pmatrix} + 3\begin{pmatrix}3\\-4\\0\end{pmatrix} = \begin{pmatrix}8\\-19\\11\end{pmatrix}$$

$$D(8,-19,11)$$

7iii) $A(0,0,6)$
$B(-1,-7,11)$
$C(-8,-6,6)$

$-3(0)+4(0)+5(6) = 30$ ✓

$-3(-1)+4(-7)+5(11)$
$= 3 - 28 + 55 = 30$ ✓

$-3(-8)+4(-6)+5(6)$
$= 24 - 24 + 30 = 30$ ✓

A, B, C on plane $-3x+4y+5z=30$

A Normal is $\begin{pmatrix}-3\\4\\5\end{pmatrix}$

iv) $\vec{BD} = 3\begin{pmatrix}3\\-4\\0\end{pmatrix}$

$$\begin{pmatrix}4\\3\\5\end{pmatrix}.\begin{pmatrix}3\\-4\\0\end{pmatrix} = 12-12+0 = 0$$

$\therefore \perp$

$$\vec{AB} = \begin{pmatrix}-1\\-7\\5\end{pmatrix}$$

$$\begin{pmatrix}4\\3\\5\end{pmatrix}.\begin{pmatrix}-1\\-7\\5\end{pmatrix} = -4-21+25 = 0$$

$\therefore \perp$

Since $\begin{pmatrix}4\\3\\5\end{pmatrix}$ is $\perp$ to two non-parallel vectors within plane ABDE, it is normal to the plane.

Plane of form $4x+3y+5z=d$

A on plane so $4(0)+3(0)+5\times6=d$

$\Rightarrow d = 30$

Plane is $4x+3y+5z=30$

7v) Angle between planes is same as angle between normals to the planes

$$\cos\theta = \frac{\begin{pmatrix}-3\\4\\5\end{pmatrix}\cdot\begin{pmatrix}4\\3\\5\end{pmatrix}}{\left|\begin{pmatrix}-3\\4\\5\end{pmatrix}\right|\left|\begin{pmatrix}4\\3\\5\end{pmatrix}\right|}$$

$$\cos\theta = \frac{-12+12+25}{\sqrt{(-3)^2+4^2+5^2}\sqrt{4^2+3^2+5^2}}$$

$$\cos\theta = \frac{25}{\sqrt{50}\sqrt{50}}$$

$$\cos\theta = \frac{25}{50} = \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ$$

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