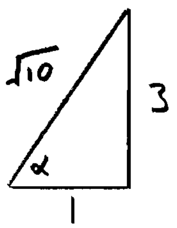
1) $\sin\theta - 3\cos\theta$



$= \sqrt{10}\left(\frac{1}{\sqrt{10}}\sin\theta - \frac{3}{\sqrt{10}}\cos\theta\right)$

$= \sqrt{10}\sin(\theta - \alpha)$

where $\alpha = \tan^{-1}\frac{3}{1} = 71.6^\circ$

$\sqrt{10}\sin(\theta - 71.6^\circ)$

$\sin\theta - 3\cos\theta = 1$

$\Rightarrow \sqrt{10}\sin(\theta - 71.6^\circ) = 1$

$\Rightarrow \sin(\theta - 71.6^\circ) = \frac{1}{\sqrt{10}}$

$\theta - 71.6^\circ = \sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$

$\theta - 71.6^\circ = 18.4^\circ, 161.6^\circ$

$\theta = 90^\circ, 233.2^\circ$

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2) Normal to $2x + 3y + 4z = 10$ is $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Normal to $x - 2y + z = 5$ is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} . \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 2 - 6 + 4 = 0$

Normals are $\perp$ so planes are $\perp$

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3) i) Volume $= \int_0^2 \pi x^2 \, dy$

Since $y = \ln x$

$e^y = x$

$\therefore$ Volume $= \int_0^2 \pi (e^y)^2 \, dy$

$= \int_0^2 \pi e^{2y} \, dy$

ii) $\int_0^2 \pi e^{2y} \, dy = \pi\left[\frac{1}{2}e^{2y}\right]_0^2$

$= \frac{\pi}{2}\left[e^4 - e^0\right]$

$= \frac{\pi}{2}\left(e^4 - 1\right)$

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4) ① $x = \frac{1}{t} - 1$, $y = \frac{2+t}{1+t}$ ②

From ① $\frac{1}{t} = x + 1$

$t = \frac{1}{x+1}$

Subst for t in ②

$$y = \frac{2 + \frac{1}{x+1}}{1 + \frac{1}{x+1}}$$

4 cont)

$$y = \frac{\frac{2(x+1)+1}{x+1}}{\frac{1(x+1)+1}{x+1}}$$

$$y = \frac{2x+3}{x+2}$$

5)

$$\underline{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\3\end{pmatrix}$$

By inspection, when $\lambda = 2$

$$r = \begin{pmatrix}1-2\\2+4\\-1+6\end{pmatrix} = \begin{pmatrix}-1\\6\\5\end{pmatrix}$$

$\therefore (-1, 6, 5)$ is on line

$$\underline{r} = \begin{pmatrix}0\\6\\3\end{pmatrix} + \mu\begin{pmatrix}1\\0\\-2\end{pmatrix}$$

By inspection, when $\lambda = -1$

$$\underline{r} = \begin{pmatrix}0-1\\6+0\\3-2(-1)\end{pmatrix} = \begin{pmatrix}-1\\6\\5\end{pmatrix}$$

$\therefore (-1, 6, 5)$ is on line

Angle between lines is angle between direction vectors

$$\cos\theta = \frac{\begin{pmatrix}-1\\2\\3\end{pmatrix}.\begin{pmatrix}1\\0\\-2\end{pmatrix}}{\left|\begin{pmatrix}-1\\2\\3\end{pmatrix}\right|\left|\begin{pmatrix}1\\0\\-2\end{pmatrix}\right|}$$

$$\cos\theta = \frac{-1+0-6}{\sqrt{(-1)^2+2^2+3^2}\sqrt{1^2+0^2+(-2)^2}}$$

$$\cos\theta = \frac{-7}{\sqrt{14}\sqrt{5}} = \frac{-7}{\sqrt{70}}$$

$$\theta = 146.8^\circ$$

Acute angle between lines

$$= 33.2^\circ$$

6)

i) $\text{Area} = \frac{h}{2}\left[y_0 + 2y_1 + y_2\right]$

$$= \frac{0.5}{2}\left[1.1696 + 2\times 1.1060 + 1.0655\right]$$

$$= 1.111775$$

$$= 1.11 \text{ to 3 sig fig}$$

ii) $(1+e^{-x})^{\frac{1}{2}}$

$$\approx 1 + \tfrac{1}{2}(e^{-x}) + \frac{\frac{1}{2}\cdot-\frac{1}{2}}{1\cdot 2}(e^{-x})^2$$

$$= 1 + \tfrac{1}{2}e^{-x} - \tfrac{1}{8}e^{-2x}$$

iii) $\displaystyle\int_1^2 \sqrt{1+e^{-x}}\,dx \approx \int_1^2 \left(1+\tfrac{1}{2}e^{-x}-\tfrac{1}{8}e^{-2x}\right)dx$

$$= \left[x - \tfrac{1}{2}e^{-x} + \tfrac{1}{16}e^{-2x}\right]_1^2$$

$$= \left(2 - \tfrac{1}{2}e^{-2} + \tfrac{1}{16}e^{-4}\right) - \left(1 - \tfrac{1}{2}e^{-1} + \tfrac{1}{16}e^{-2}\right)$$

$$= 1.933477 - 0.824519$$

$$= 1.108958 = 1.11 \text{ to 3 sig fig}$$

7) a) $P = \dfrac{2}{2 - \sin t}$ $\quad t=0 \Rightarrow P=1$

i) $\text{Max } P = \dfrac{2}{2-1} = 2$

$\text{Min } P = \dfrac{2}{2-(-1)} = \dfrac{2}{3}$

ii) $P = \dfrac{2}{2-\sin t}$

$\dfrac{dP}{dt} = \dfrac{(2-\sin t)0 - 2(-\cos t)}{(2-\sin t)^2}$

$\dfrac{dP}{dt} = \dfrac{2\cos t}{(2-\sin t)^2}$

$\dfrac{dP}{dt} = \dfrac{1}{2} \times \dfrac{4}{(2-\sin t)^2} \cos t$

$\dfrac{dP}{dt} = \dfrac{1}{2} P^2 \cos t$

$\therefore$ P satisfies this diff. eqn.

7b) $\dfrac{dP}{dt} = \dfrac{1}{2}(2P^2 - P)\cos t$

$P=1$ when $t=0$

i) Let $\dfrac{1}{P(2P-1)} \equiv \dfrac{A}{P} + \dfrac{B}{2P-1}$

$\Rightarrow 1 \equiv A(2P-1) + BP$

When $P=0$

$1 = -A \quad \Rightarrow A = -1$

When $P = \frac{1}{2}$

$1 = \frac{1}{2}B \quad \Rightarrow B = 2$

$\dfrac{1}{P(2P-1)} \equiv \dfrac{2}{2P-1} - \dfrac{1}{P}$

ii) $\displaystyle\int \frac{1}{2P^2 - P}\, dP = \int \frac{1}{2}\cos t \, dt$

$\displaystyle\int \left(\frac{2}{2P-1} - \frac{1}{P}\right) dP = \int \frac{1}{2}\cos t \, dt$

$\ln(2P-1) - \ln P = \frac{1}{2}\sin t + c$

$\ln\left(\dfrac{2P-1}{P}\right) = \frac{1}{2}\sin t + c$

$P=1$, when $t=0$ so

$\ln\left(\dfrac{1}{1}\right) = \frac{1}{2}\sin 0 + c$

$0 = 0 + c \quad \Rightarrow c = 0$

$\therefore \ln\left(\dfrac{2P-1}{P}\right) = \frac{1}{2}\sin t$

iii) $P = \dfrac{1}{2 - e^{\frac{1}{2}\sin t}}$

$\text{Min } P = \dfrac{1}{2 - e^{-0.5}}$

$= 0.718$ to 3 d.p

$\text{Max } P = \dfrac{1}{2 - e^{0.5}}$

$= 2.847$ to 3 d.p

8) $x = 10\cos\theta + 5\cos 2\theta$
$y = 10\sin\theta + 5\sin 2\theta$
$(0 \le \theta < 2\pi)$

i) $\frac{dy}{d\theta} = 10\cos\theta + 10\cos 2\theta$

$\frac{dx}{d\theta} = -10\sin\theta - 10\sin 2\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{10\cos\theta + 10\cos 2\theta}{-10\sin\theta - 10\sin 2\theta}$

$\frac{dy}{dx} = -\frac{\cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}$

when $\theta = \frac{\pi}{3}$

$\frac{dy}{dx} = -\frac{\cos\frac{\pi}{3} + \cos\frac{2\pi}{3}}{\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}}$

$= -\left(\frac{\frac{1}{2} + -\frac{1}{2}}{\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}}\right)$

$= 0$

At A, $\frac{dy}{dx} = 0$

$x = 10\cos\frac{\pi}{3} + 5\cos\frac{2\pi}{3}$

$x = 10 \times \frac{1}{2} + 5(-\frac{1}{2}) = 5/2$

$y = 10\sin\frac{\pi}{3} + 5\sin\frac{2\pi}{3}$

$y = 10 \times \frac{\sqrt{3}}{2} + 5 \times \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{2}$

A is point $\left(\frac{5}{2}, \frac{15\sqrt{3}}{2}\right)$

8ii) $x^2 + y^2 =$

$(10\cos\theta + 5\cos 2\theta)^2 + (10\sin\theta + 5\sin 2\theta)^2$

$= 100\cos^2\theta + 100\cos\theta\cos 2\theta + 25\cos^2 2\theta$
$+ 100\sin^2\theta + 100\sin\theta\sin 2\theta + 25\sin^2 2\theta$

$= 100(\cos^2\theta + \sin^2\theta) + 25(\cos^2 2\theta + \sin^2 2\theta)$
$+ 100(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta)$

$= 100 + 25 + 100\cos(2\theta - \theta)$

$= 125 + 100\cos\theta$

8iii) $(\text{Distance from } O)^2 = x^2 + y^2$

Max Dist^2 = 125 + 100 = 225

Max Distance = 15 m

Min Dist^2 = 125 − 100 = 25

Min Distance = 5 m

8iv) $2\cos^2\theta + 2\cos\theta - 1 = 0$

By calculator $\cos\theta = 0.3660254$
$\cos\theta = -1.3660254$

Now $x^2 + y^2 = 125 + 100\cos\theta$
and at B, $x = 0$

$\therefore y^2 = 125 + 100 \times 0.3660254$

$\Rightarrow y^2 = 161.60254$

$\Rightarrow y = 12.712$

$\therefore$ OB = 12.7 m to 3 s.f.

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