MEI CORE 3 INVERSE FUNCTIONS REVISION

1) $\sin^{-1} x = \frac{\pi}{6}$

Find $\cos^{-1} x$ in terms of π

$$x = \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{3}$$

2) $f(x) = 1 + 2\sin x$

$-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$

i) Let $y = 1 + 2\sin x$

Swap variables $x = 1 + 2\sin y$

Rearrange to make y subject

$$x = 1 + 2\sin y$$

$$x - 1 = 2\sin y$$

$$\frac{x-1}{2} = \sin y$$

$$y = \sin^{-1}\left(\frac{x-1}{2}\right)$$

$$\therefore\ f^{-1}(x) = \sin^{-1}\left(\frac{x-1}{2}\right)$$

Domain of $f^{-1}(x)$ is the range of $f(x)$

When $x = -\frac{\pi}{2}$, $f(x) = 1 + 2\sin\left(-\frac{\pi}{2}\right)$

$= 1 - 2$

$= -1$

When $x = \frac{\pi}{2}$, $f(x) = 1 + 2\sin\frac{\pi}{2}$

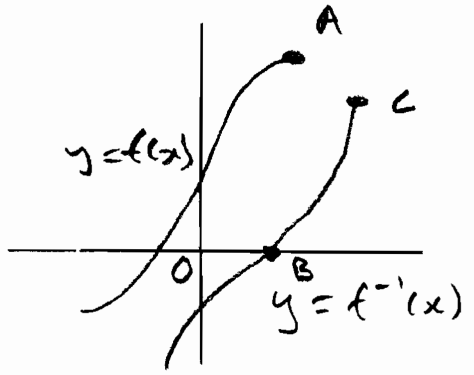
$= 1 + 2 = 3$

Range of $f(x)$

$$-1 \le f(x) \le 3$$

$\therefore$ domain of $f^{-1}(x)$ is

$$1 \le x \le 3$$



$A\left(\frac{\pi}{2}, 3\right)$

$B(1, 0)$

$C\left(3, \frac{\pi}{2}\right)$

Find B by considering corresponding point on $y = f(x)$

$$y = f(0) = 1 + 2\sin 0 = 1$$

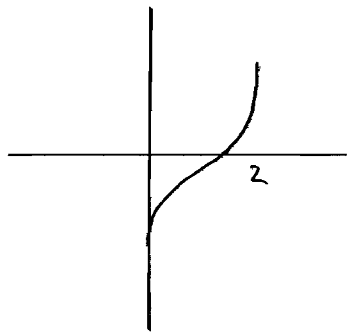
so $(0, 1)$ on $f(x)$

becomes $(1, 0)$ on $f^{-1}(x)$

Likewise A and C correspond so simply swap coordinates round

$\left(\frac{\pi}{2}, 3\right)$ becomes $\left(3, \frac{\pi}{2}\right)$

3) $y = \sin^{-1}(x-1) \quad 0 \leq x \leq 2$



i) $\sin y = x - 1$

$\therefore x = \sin y + 1$

$\frac{dx}{dy} = \cos y$

ii) When $x = 1.5$

$y = \sin^{-1}(1.5 - 1)$

$y = \sin^{-1}(\frac{1}{2})$

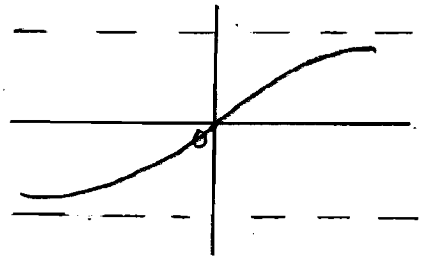
$y = \frac{\pi}{6}$

$\therefore \frac{dx}{dy} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\therefore$ gradient $= \frac{dy}{dx} = \frac{2}{\sqrt{3}}$

or $\frac{dy}{dx} = \frac{2\sqrt{3}}{3}$

4) $f(x) = \frac{1}{2}\tan^{-1}x$



i) $\tan^{-1}\infty = \frac{\pi}{2}$

so $f(x) \to \frac{1}{2} \times \frac{\pi}{2}$ as $x \to \infty$

$\tan^{-1}(-\infty) = -\frac{\pi}{2}$

so $f(x) \to \frac{1}{2}(-\frac{\pi}{2})$ as $x \to -\infty$

Range $-\frac{\pi}{4} < f(x) < \frac{\pi}{4}$

ii) Let $y = \frac{1}{2}\tan^{-1}x$

swap variables $x = \frac{1}{2}\tan^{-1}y$

Rearrange to make y the subject

$2x = \tan^{-1}y$

$\tan(2x) = y$

$f^{-1}(x) = \tan 2x$

$\frac{d f^{-1}(x)}{dx} = 2\sec^2 2x$

$= \frac{2}{\cos^2 2x}$

when $x = 0$, gradient $= \frac{2}{\cos^2 0} = 2$

4 ii) (cont) So gradient of $f^{-1}(x)$ at origin $= 2$

iii) Gradient of $f(x) = \frac{1}{2}$

(Since gradients of inverse functions are positive reciprocals of each other at corresponding points)

5) $f(x) = \frac{1}{2}\ln(x-1)$

$g(x) = 1 + e^{2x}$

Show $g(x)$ is inverse of $f(x)$

$f(x) = \frac{1}{2}\ln(x-1)$

Let $y = \frac{1}{2}\ln(x-1)$

Swap variables

$x = \frac{1}{2}\ln(y-1)$

Rearrange to make y subject

$$2x = \ln(y-1)$$
$$e^{2x} = e^{\ln(y-1)}$$
$$e^{2x} = y-1$$
$$e^{2x}+1 = y$$
$$\therefore f^{-1}(x) = 1 + e^{2x} = g(x)$$

6) Sketch $y = 2\cos^{-1}x$ for $-1 \le x \le 1$

$x = -1$, $y = 2\cos^{-1}(-1) = 2(\pi) = 2\pi$

$x = 0$, $y = 2\cos^{-1}0 = 2 \times \left(\frac{\pi}{2}\right) = \pi$

$x = 1$, $y = 2\cos^{-1}1 = 0$