1) i) $\frac{dy}{dx} = x^2 - 1$ $y = 2$ when $x = 3$

$y = \frac{x^3}{3} - x + c$

$2 = \frac{27}{3} - 3 + c$

$2 = 6 + c$

$-4 = c$

$y = \frac{x^3}{3} - x - 4$

ii)

$\frac{dy}{dx} = x^2 y$ $y = 1$ when $x = 0$

$\int \frac{1}{y}\,dy = \int x^2\,dx$

$\ln y = \frac{x^3}{3} + c$

$\ln 1 = 0 + c$

$0 = c$

$\ln y = \frac{x^3}{3}$

$y = e^{\frac{x^3}{3}}$

iii)

$\frac{dy}{dx} = x e^{-y}$ $y = 0$ when $x = 0$

$\frac{dy}{dx} = \frac{x}{e^y}$

$\int e^y\,dy = \int x\,dx$

$e^y = \frac{x^2}{2} + c$

$e^0 = 0 + c$

$1 = c$

$e^y = \frac{x^2}{2} + 1$

$y = \ln\left(\frac{x^2+2}{2}\right)$

iv)

$\frac{dy}{dx} = y^2$ $y = 1$ when $x = 1$

$\int \frac{1}{y^2}\,dy = \int 1\,dx$

$-\frac{1}{y} = x + c$

$-\frac{1}{1} = 1 + c$

$-2 = c$

$\therefore \quad -\frac{1}{y} = x - 2$

$2 - x = \frac{1}{y}$

$y = \frac{1}{2-x}$

v)

$\frac{dy}{dx} = x(y+1)$ $y = 0$ when $x = 1$

$\int \frac{1}{y+1}\,dy = \int x\,dx$

$\ln(y+1) = \frac{x^2}{2} + c$

$\ln(1) = \frac{1}{2} + c$

1v) cont)

$$0 = \frac{1}{2} + c$$
$$-\frac{1}{2} = c$$
$$\ln(y+1) = \frac{x^2}{2} - \frac{1}{2}$$
$$\ln(y+1) = \frac{x^2-1}{2}$$
$$y+1 = e^{\left(\frac{x^2-1}{2}\right)}$$
$$y = e^{\left(\frac{x^2-1}{2}\right)} - 1$$

vi)

$$\frac{dy}{dx} = y^2 \sin x \qquad y = 1 \text{ when } x = 0$$
$$\int \frac{1}{y^2}\,dy = \int \sin x\,dx$$
$$-\frac{1}{y} = -\cos x + c$$
$$-\frac{1}{1} = -1 + c$$
$$\Rightarrow c = 0$$
$$-\frac{1}{y} = -\cos x$$
$$\frac{1}{y} = \cos x$$
$$y = \frac{1}{\cos x}$$
$$\text{or } y = \sec x$$

2)

$$\frac{d\theta}{dt} = 2(20-\theta)$$

i)

$$\int \frac{1}{20-\theta}\,d\theta = \int 2\,dt$$
$$-\ln(20-\theta) = 2t + c$$
$$\ln(20-\theta) = -2t + c$$
$$20-\theta = e^{-2t+c}$$
$$20-\theta = Ae^{-2t}$$
$$20 - Ae^{-2t} = \theta$$
$$\theta = 20 - Ae^{-2t}$$

2ii)

$$\theta = 5 \text{ when } t = 0$$
$$5 = 20 - Ae^0$$
$$5 = 20 - A$$
$$\Rightarrow A = 15$$
$$\theta = 20 - 15e^{-2t}$$

2iii)

$$18 = 20 - 15e^{-2t}$$
$$15e^{-2t} = 20 - 18$$
$$e^{-2t} = \frac{2}{15}$$
$$-2t = \ln\left(\frac{2}{15}\right)$$
$$t = \frac{\ln\left(\frac{2}{15}\right)}{-2}$$

$t = 1.01$ hours to 3 sig fig

3) $\frac{dN}{dt} = N$

i) $\int \frac{1}{N}\, dN = \int 1\, dt$

$\ln N = t + c$

$N = e^{t+c}$

$N = Ae^{t}$

3ii) $N = 10$ when $t = 0$

$10 = Ae^{0}$

$\Rightarrow A = 10$

$N = 10e^{t}$

3iii) $N \to \infty$ as $t \to \infty$

Rabbit population cannot have unlimited growth due to food supplies, predators etc.

4)

i) $\frac{ds}{dt} = \frac{k}{s}$

Given $\frac{ds}{dt} = 1$ when $s = 2$

$\therefore 1 = \frac{k}{2}$

$\Rightarrow k = 2$

Differential eqn is

$\frac{ds}{dt} = \frac{2}{s}$

$\int s\, ds = \int 2\, dt$

$\frac{s^2}{2} = 2t + c$

$s^2 = 4t + c$

$s = \sqrt{4t+c}$

5)

i) Let $\frac{1}{y(3-y)} \equiv \frac{A}{y} + \frac{B}{(3-y)}$

Then

$1 \equiv A(3-y) + By$

When $y = 0$ $\quad 1 = 3A \Rightarrow A = \frac{1}{3}$

when $y = 3$ $\quad 1 = 3B \Rightarrow B = \frac{1}{3}$

$\therefore \frac{1}{y(3-y)} \equiv \frac{1}{3y} + \frac{1}{3(3-y)}$

5ii) $\int \frac{1}{y(3-y)}\, dy = \int \left(\frac{1}{3y} + \frac{1}{3(3-y)}\right) dy$

$= \frac{1}{3}\ln y - \frac{1}{3}\ln(3-y) + c$

$= \frac{1}{3}\ln\left|\frac{y}{3-y}\right| + c$

5iii) $x\frac{dy}{dx} = y(3-y)$

where $x = 2$ when $y = 2$

5 iii) cont

$$\int \frac{1}{y(3-y)}\,dy = \int \frac{1}{x}\,dx$$

$$\frac{1}{3}\ln\left|\frac{y}{3-y}\right| = \ln x + c$$

$x = 2$, when $y = 2$

$$\frac{1}{3}\ln\left|\frac{2}{1}\right| = \ln 2 + c$$

$$\Rightarrow c = -\frac{2}{3}\ln 2$$

$$\therefore \frac{1}{3}\ln\left|\frac{y}{3-y}\right| = \ln x - \frac{2}{3}\ln 2$$

$$\ln\left|\frac{y}{3-y}\right| = 3\ln x - 2\ln 2$$

$$\ln\left|\frac{y}{3-y}\right| = \ln x^3 - \ln 4$$

$$\ln\left|\frac{y}{3-y}\right| = \ln\left|\frac{x^3}{4}\right|$$

$$\frac{y}{3-y} = \frac{x^3}{4}$$

$$4y = x^3(3-y)$$

$$4y = 3x^3 - x^3y$$

$$4y + x^3y = 3x^3$$

$$y(4+x^3) = 3x^3$$

$$y = \frac{3x^3}{4+x^3}$$

6) $$\frac{dy}{dt} + ky = 2k$$

$$\frac{dy}{dt} = 2k - ky = k(2-y)$$

$$\int \frac{1}{2-y}\,dy = \int k\,dt$$

$$-\ln|2-y| = kt + c$$

Given $y = 3$ when $t = 0$

$$-\ln|-1| = 0 + c$$

$$\Rightarrow c = 0$$

$$-\ln|2-y| = kt$$

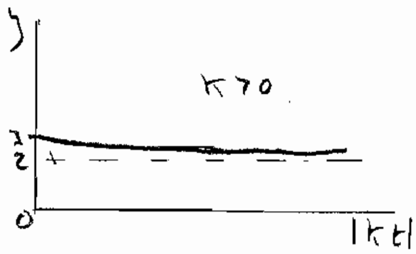
$$\ln|2-y| = -kt$$

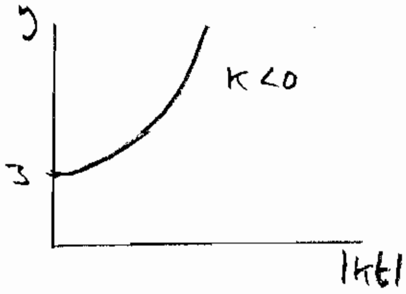
$$|2-y| = e^{-kt}$$

for $y > 2$

$$y - 2 = e^{-kt}$$

$$y = 2 + e^{-kt}$$





7) $\frac{dN}{dt} = kN$

i) $\int \frac{1}{N}\,dN = \int k\,dt$

$\ln N = kt + c$

$N = e^{kt+c}$

$N = Ae^{kt}$

$N = 1500$ when $t = 0$

$1500 = Ae^{0}$

$\Rightarrow A = 1500$

$N = 1500e^{kt}$

$N = 3000$ when $t = 20$

$3000 = 1500e^{20k}$

$2 = e^{20k}$

$\ln 2 = 20k$

$k = \frac{\ln 2}{20}$

$k = 0.0347$

$\therefore N = 1500e^{0.0347t}$

ii) When $t = 80$

$N = 1500e^{0.0347 \times 80}$

$N = 24082 \approx 24{,}000$

$N = 3000$ when $t = 20$

When $N = 2000$

$2000 = 1500e^{0.0347t}$

$\frac{4}{3} = e^{0.0347t}$

$\ln\left(\frac{4}{3}\right) = 0.0347t$

$t = \frac{\ln\left(\frac{4}{3}\right)}{0.0347}$

$t = 8.29$ hrs

Time taken between $N = 2000$ and $N = 3000$

$= 20 - 8.29055$ hrs

$= 11.70945$ hrs

$= 11$ hrs 43 mins

8) i) $\frac{x^2+1}{x^2-1}$

$$\begin{array}{r} 1 \\ x^2-1 \overline{\smash{)}\, x^2+1} \\ \underline{x^2-1} \\ 2 \end{array}$$

$= 1 + \frac{2}{x^2-1}$

ii) Let $\frac{2}{(x-1)(x+1)} \equiv \frac{A}{x-1} + \frac{B}{x+1}$

$\Rightarrow 2 \equiv A(x+1) + B(x-1)$

8ii) cont) when $x=1$

$$2 = 2A \Rightarrow A = 1$$

when $x=-1$

$$2 = -2B \Rightarrow B = -1$$

$$\frac{2}{(x-1)(x+1)} \equiv \frac{1}{x-1} - \frac{1}{x+1}$$

iii)

$$(x^2-1)\frac{dy}{dx} = -(x^2+1)y$$

$$\int -\frac{1}{y}\,dy = \int \frac{(x^2+1)}{(x^2-1)}\,dx$$

$$\int -\frac{1}{y}\,dy = \int \left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$-\ln y = x + \ln(x-1) - \ln(x+1) + c$$

When $x=3$, $y=1$

$$\therefore -\ln 1 = 3 + \ln 2 - \ln 4 + c$$

$$0 = 3 + \ln\left(\tfrac{1}{2}\right) + c$$

$$c = -3 - \ln\left(\tfrac{1}{2}\right)$$

$$\therefore -\ln y = x + \ln(x-1) - \ln(x+1) - 3 - \ln\left(\tfrac{1}{2}\right)$$

$$3 - x = \ln(x-1) - \ln(x+1) + \ln y - \ln\left(\tfrac{1}{2}\right)$$

$$3 - x = \ln\left[\frac{(x-1)y}{\frac{1}{2}(x+1)}\right]$$

$$3 - x = \ln\left[\frac{2(x-1)y}{(x+1)}\right]$$

$$\frac{2(x-1)y}{(x+1)} = e^{3-x}$$

$$y = \frac{(x+1)e^{3-x}}{2(x-1)}$$

9)

$$\frac{dV}{dt} = -kx$$

$$V = \pi\left(ax^2 - \tfrac{1}{3}x^3\right)$$

when $t=0$, $x=a$

$$\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt} = \frac{1}{\frac{dV}{dx}} \times \frac{dV}{dt}$$

$$\frac{dx}{dt} = \frac{1}{\pi(2ax-x^2)} \times -kx$$

$$\therefore \pi(2ax-x^2)\frac{dx}{dt} = -kx$$

At $t=T$, $x=0$

$$\pi\int \frac{(2ax-x^2)}{x}\,dx = \int -k\,dt$$

$$\pi\int (2a-x)\,dx = \int -k\,dt$$

$$\pi\left[2ax - \frac{x^2}{2}\right] = -kt + c$$

At $t=0$, $x=a$

9 (cont)

$$\pi\left[2a^2 - \frac{a^2}{2}\right] = 0 + c$$

$$\Rightarrow c = \frac{3\pi a^2}{2}$$

$$\therefore \pi\left[2ax - \frac{x^2}{2}\right] = -kt + \frac{3\pi a^2}{2}$$

$x = 0$, when $t = T$

$$\therefore \pi[0 - 0] = -kT + \frac{3\pi a^2}{2}$$

$$\Rightarrow kT = \frac{3\pi a^2}{2}$$

$$k = \frac{3\pi a^2}{2T}$$

10) $V = 4x$, $\frac{dV}{dt} = 0.08 - 0.12x$

$$\frac{dx}{dt} = \frac{dx}{dV} \cdot \frac{dV}{dt}$$

$$= \frac{1}{\frac{dV}{dx}} \times \frac{dV}{dt}$$

$$\frac{dx}{dt} = \frac{1}{4}(0.08 - 0.12x)$$

$$\frac{dx}{dt} = 0.02 - 0.03x$$

$$\int \frac{1}{(0.02 - 0.03x)}\,dx = \int 1\,dt$$

$$-\frac{1}{0.03}\ln(0.02 - 0.03x) = t + c$$

$x = 0$ when $t = 0$

$$\therefore -\frac{1}{0.03}\ln(0.02) = 0 + c$$

$$\Rightarrow c = 130.4$$

$$\therefore -\frac{1}{0.03}\ln(0.02 - 0.03x) = t + 130.4$$

$$\ln(0.02 - 0.03x) = -0.03t - 3.912$$

$$0.02 - 0.03x = e^{-0.03t - 3.912}$$

$$0.02 - 0.03x = 0.02e^{-0.03t}$$

$$0.02 - 0.02e^{-0.03t} = 0.03x$$

$$x = \frac{0.02 - 0.02e^{-0.03t}}{0.03}$$

$$x = \frac{1}{3}\left(2 - 2e^{-0.03t}\right)$$

$$x = 0.1 \Rightarrow 0.1 = \frac{1}{3}\left(2 - 2e^{-0.03t}\right)$$

$$0.3 = 2 - 2e^{-0.03t}$$

$$2e^{-0.03t} = 1.7$$

$$-0.03t = \ln\left(\frac{1.7}{2}\right)$$

$$t = \frac{\ln\left(\frac{1.7}{2}\right)}{-0.03}$$

$$t = 5.417 \text{ s}$$

10 cont) $x = 0.5$

$$\Rightarrow \quad 0.5 = \frac{1}{3}\left(2 - 2e^{-0.03t}\right)$$

$$1.5 = 2 - 2e^{-0.03t}$$

$$2e^{-0.03t} = 0.5$$

$$e^{-0.03t} = 0.25$$

$$-0.03t = \ln(0.25)$$

$$t = \frac{\ln(0.25)}{-0.03}$$

$$t = 46.210 \text{ s}$$

Time taken for x to rise from 0.1 to 0.5 m

$$= 46.210 - 5.417$$

$$= 40.8 \text{ s} \quad \text{to 1 dp.}$$

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11)

i) $\dfrac{dP}{dt} = kP$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln P = kt + c$$

$$P = e^{kt+c}$$

$$P = Ae^{kt}$$

$P = 600$ when $t = 0$

$$\Rightarrow \quad P = 600e^{kt}$$

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11 ii) $P = 1200$ when $t = 150$

$$1200 = 600e^{150k}$$

$$\Rightarrow \quad e^{150k} = 2$$

$$150k = \ln 2$$

$$k = \frac{\ln 2}{150} \approx 0.00462$$

If $P = 600e^{0.00462t}$

when $t = 250$

$$P = 600e^{0.00462 \times 250}$$

$$P = 1904$$

Not consistent with observed $P = 3100$

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11 iii)

$$\frac{dP}{dt} = P\left[0.005 - 0.008\cos(0.02t)\right]$$

$$\frac{dP}{dt} = \frac{P}{1000}\left[5 - 8\cos(0.02t)\right]$$

$$\int \frac{1000}{P}\,dP = \int 5 - 8\cos(0.02t)\,dt$$

$$1000\ln P = 5t - \frac{8\sin(0.02t)}{0.02} + c$$

$P = 600$ when $t = 0$

$$\Rightarrow \quad 1000\ln 600 = c$$

$$1000\ln P = 5t - 400\sin(0.02t) + 1000\ln 600$$

11 iii) cont

$\ln P = 0.005t - 0.4\sin(0.02t) + \ln 600$

$\ln\left(\frac{P}{600}\right) = 0.005t - 0.4\sin(0.02t)$

$\frac{P}{600} = e^{(0.005t - 0.4\sin(0.02t))}$

$P = 600e^{(0.005t - 0.4\sin(0.02t))}$

When $t = 150$

$P = 600e^{(0.75 - 0.4\sin 3)}$

$P = 1200.49 \approx 1200$

when $t = 250$

$P = 600e^{(1.25 - 0.4\sin 5)}$

$P = 3073 \approx 3100$

A very good fit with observed data

iv) At $t = 0$

$\frac{dP}{dt} = 600\left[0.005 - 0.008\cos 0\right]$

$= 600\left[-0.003\right]$

$\frac{dP}{dt} = -1.8$

$\therefore$ P decreasing initially

Minimum when $\frac{dP}{dt} = 0$

$\Rightarrow 0.005 - 0.008\cos(0.02t) = 0$

$\Rightarrow 0.005 = 0.008\cos(0.02t)$

$\frac{0.005}{0.008} = \cos(0.02t)$

$0.02t = \cos^{-1}\left(\frac{5}{8}\right)$

$\Rightarrow t = 44.7832$

At this time

$P = 600e^{(0.223916 - 0.312250)}$

$P = 549.27$

$P = 549$ ladybirds