1) i) $(1+x)^{-2}$

$\approx 1+(-2)x + \frac{(-2)(-3)}{1.2}x^2$

$= 1 - 2x + 3x^2$

Valid for $|x|<1$

$(1+0.1)^{-2} = 0.826446281$

$1-2(0.1)+3(0.1)^2 = 0.83$

relative error 0.43%

ii) $\frac{1}{1+2x} = (1+2x)^{-1}$

$\approx 1+(-1)(2x)+\frac{(-1)(-2)}{1.2}(2x)^2$

$= 1-2x+4x^2$

Valid for $|2x|<1 \Rightarrow |x|<\frac{1}{2}$

$\frac{1}{1+0.2} = 0.833333333$

$1-2(0.1)+4(0.1)^2 = 0.84$

relative error $= 0.8\%$

iii) $\sqrt{1-x^2} = (1-x^2)^{\frac{1}{2}}$

$\approx 1+\frac{1}{2}(-x^2)+\frac{\frac{1}{2}\cdot-\frac{1}{2}}{1.2}(-x^2)^2$

$= 1-\frac{1}{2}x^2-\frac{1}{8}x^4$

Valid for $|x^2|<1 \Rightarrow |x|<1$

$\sqrt{0.99} = 0.9949874371$

$1-\frac{1}{2}(0.1)^2-\frac{1}{8}(0.1)^4 = 0.9949875$

relative error $= 0.0000063\%$

iv) $\frac{1+2x}{1-2x} = (1+2x)(1-2x)^{-1}$

$(1-2x)^{-1} \approx 1+(-1)(-2x)+\frac{(-1)(-2)(-2x)^2}{1.2}$

$= 1+2x+4x^2$

$(1+2x)(1-2x)^{-1}$

$\approx (1+2x)(1+2x+4x^2)$

$\approx 1+2x+2x+4x^2+4x^2$

$= 1+4x+8x^2$

Valid for $|2x|<1 \Rightarrow |x|<\frac{1}{2}$

$\frac{1.2}{0.8} = 1.5$

$1+4(0.1)+8(0.1)^2 = 1.48$

relative error $= 1.33\%$

v) $(3+x)^{-1} = \frac{1}{3}\left(1+\frac{x}{3}\right)^{-1}$

$\approx \frac{1}{3}\left(1+(-1)\left(\frac{x}{3}\right)+\frac{(-1)(-2)}{1.2}\left(\frac{x}{3}\right)^2\right)$

$= \frac{1}{3}\left(1-\frac{x}{3}+\frac{x^2}{9}\right)$

$= \frac{1}{3}-\frac{x}{9}+\frac{x^2}{27}$ Valid for $|x|<3$

1(v) cont

$$\frac{1}{3.1} = 0.3225806452$$

$$\frac{1}{3} - \frac{0.1}{9} + \frac{0.1^2}{27} = 0.3225925926$$

relative error = 0.0037 %

1(vi)

$$(1-x)\sqrt{4+x} = (1-x)2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

$$\approx (1-x)2\left[1 + \frac{1}{2}\left(\frac{x}{4}\right) + \frac{\frac{1}{2}\cdot -\frac{1}{2}}{1.2}\left(\frac{x}{4}\right)^2\right]$$

$$= (2-2x)\left[1 + \frac{x}{8} - \frac{x^2}{128}\right]$$

$$\approx 2 - 2x + \frac{x}{4} - \frac{x^2}{4} - \frac{x^2}{64}$$

$$= 2 - \frac{7x}{4} - \frac{17x^2}{64}$$

Valid for $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$

$0.9\sqrt{4.1} = 1.822361106$

$$2 - \frac{7(0.1)}{4} - \frac{17(0.1)^2}{64} = 1.82234375$$

relative error = 0.00095 %

1(vii)

$$\frac{x+2}{x-3} = (x+2)(x-3)^{-1}$$

$$= -(x+2)(3-x)^{-1}$$

$$= -\frac{1}{3}(x+2)\left(1-\frac{x}{3}\right)^{-1}$$

$$\approx -\frac{1}{3}(x+2)\left[1 + (-1)\left(-\frac{x}{3}\right) + \frac{(-1)(-2)}{1.2}\left(-\frac{x}{3}\right)^2\right]$$

$$= -\frac{1}{3}(x+2)\left[1 + \frac{x}{3} + \frac{x^2}{9}\right]$$

$$\approx -\frac{1}{3}\left[x + 2 + \frac{x^2}{3} + \frac{2x}{3} + \frac{2x^2}{9}\right]$$

$$= -\frac{1}{3}\left[2 + \frac{5x}{3} + \frac{5x^2}{9}\right]$$

$$= -\frac{2}{3} - \frac{5x}{9} - \frac{5x^2}{27}$$

Valid for $\left|\frac{x}{3}\right| < 1 \Rightarrow |x| < 3$

$$\frac{2.1}{-2.9} = -0.724137931$$

$$-\frac{2}{3} - \frac{5(0.1)}{9} - \frac{5(0.1)^2}{27}$$

$$= 0.7240740741$$

relative error = 0.0088

1(viii)

$$\frac{1}{\sqrt{3x+4}} = \frac{1}{2\sqrt{1+\frac{3x}{4}}}$$

$$= \frac{1}{2}\left(1 + \frac{3x}{4}\right)^{-\frac{1}{2}}$$

$$\approx \frac{1}{2}\left[1 + -\frac{1}{2}\left(\frac{3x}{4}\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{1.2}\left(\frac{3x}{4}\right)^2\right]$$

$$= \frac{1}{2}\left[1 - \frac{3x}{8} + \frac{27x^2}{128}\right]$$

$$= \frac{1}{2} - \frac{3x}{16} + \frac{27x^2}{256}$$

Valid for $\left|\frac{3x}{4}\right| < 1$

$$\Rightarrow |x| < \frac{4}{3}$$

$$\frac{1}{\sqrt{4.3}} = 0.4822428222$$

1 viii) cont)

$$\frac{1}{2} - \frac{3}{16}(0.1) + \frac{27(0.1)^2}{256}$$

$= 0.4823046875$

relative error $= 0.013\ \%$

1 ix)

$$\frac{1+2x}{(2x-1)^2} = \frac{1+2x}{(1-2x)^2}$$

$= (1+2x)(1-2x)^{-2}$

$\approx (1+2x)\left[1 + (-2)(-2x) + \frac{(-2)(-3)(-2x)^2}{1.2}\right]$

$= (1+2x)\left[1 + 4x + 12x^2\right]$

$\approx 1 + 2x + 4x + 8x^2 + 12x^2$

$= 1 + 6x + 20x^2$

Valid for $|2x| < 1$

$\Rightarrow |x| < \frac{1}{2}$

$$\frac{1.2}{(-0.8)^2} = 1.875$$

$1 + 0.6 + 0.2 = 1.8$

relative error $= 4\ \%$

1 x)

$$\frac{1+x^2}{1-x^2} = (1+x^2)(1-x^2)^{-1}$$

$\approx (1+x^2)\left(1 + (-1)(-x^2) + \frac{(-1)(-2)(-x^2)^2}{1.2}\right)$

$= (1+x^2)(1 + x^2 + x^4)$

$\approx 1 + x^2 + x^2 + x^4 + x^4$

$= 1 + 2x^2 + 2x^4$

Valid for $|x^2| < 1$

$\Rightarrow |x| < 1$

$$\frac{1.01}{0.99} = 1.02020202$$

$1 + 0.02 + 0.0002 = 1.0202$

Relative error $= 0.0002\ \%$

1 xi)

$\sqrt[3]{1+2x^2} = (1+2x^2)^{\frac{1}{3}}$

$\approx 1 + \frac{1}{3}(2x^2) + \frac{\frac{1}{3}\cdot -\frac{2}{3}}{1.2}(2x^2)^2$

$= 1 + \frac{2}{3}x^2 - \frac{4}{9}x^4$

Valid for $|2x^2| < 1 \Rightarrow |x^2| < \frac{1}{2}$

$\Rightarrow |x| < \frac{1}{\sqrt{2}}$

$\sqrt[3]{1+2(0.1)^2} = 1.00662271$

$1 + \frac{2}{3}(0.1)^2 - \frac{4}{9}(0.1)^4 = 1.006622222$

Relative error $0.000048\ \%$

1 xii)

$$\frac{1}{(1+2x)(1+x)}$$

$= (1+2x)^{-1}(1+x)^{-1}$

$(1+2x)^{-1} \approx 1 + -1(2x) + \frac{-1.-2}{1.2}(2x)^2$

1xii cont

$= 1 - 2x + 4x^2$

$(1+x)^{-1} \approx 1 - x + \frac{-1.-2}{1.2}x^2$

$= 1 - x + x^2$

$(1-2x+4x^2)(1-x+x^2)$

$\approx 1 - 2x + 4x^2 - x + 2x^2 + x^2$

$= 1 - 3x + 7x^2$

Valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

$\frac{1}{1.2 \times 1.1} = 0.7575757575756$

$1 - 0.3 + 0.07 = 0.77$

Relative error = 1.64%

2i) $(1+x)^3 = 1 + 3x + 3x^2 + x^3$

ii) $(1-x)^{-4} \approx 1 - 4(-x) + \frac{-4.-5}{1.2}(-x)^2 + \frac{-4.-5.-6}{1.2.3}(-x)^3$

$= 1 + 4x + 10x^2 + 20x^3$

Valid for $|x| < 1$

iii) $\frac{(1+x)^3}{(1-x)^4} = (1+x)^3(1-x)^{-4}$

$\approx (1+3x+3x^2+x^3)(1+4x+10x^2+20x^3)$

$\approx 1 + 3x + 3x^2 + x^3 + 4x + 12x^2 + 12x^3 + 10x^2 + 30x^3 + 20x^3$

$= 1 + 7x + 25x^2 + 63x^3$

$\Rightarrow a = 25, \ b = 63$

3) i) $(2-x)^4$

$= 2^4 + 4\times2^3(-x) + 6\times2^2(-x)^2 + 4\times2(-x)^3 + (-x)^4$

$= 16 - 32x + 24x^2 - 8x^3 + x^4$

ii) $(1+2x)^{-3}$

$\approx 1 - 3(2x) + \frac{-3.-4}{1.2}(2x)^2 + \frac{-3.-4.-5}{1.2.3}(2x)^3$

$= 1 - 6x + 24x^2 - 80x^3$

Valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

iii) $\frac{(2-x)^4}{(1+2x)^3} = (2-x)^4(1+2x)^{-3}$

$\approx (16 - 32x + 24x^2)(1 - 6x + 24x^2)$

$\approx 16 - 32x + 24x^2 - 96x + 192x^2 + 384x^2$

$= 16 - 128x + 600x^2$

$\Rightarrow a = -128, \ b = 600$

4) i) $(1-x)^{-1}$

$\approx 1 - 1(-x) + \frac{-1.-2}{1.2}(-x)^2 + \frac{-1.-2.-3}{1.2.3}(-x)^3$

$= 1 + x + x^2 + x^3$

valid for $|x| < 1$

4ii) $(1+2x)^{-2}$

$\approx 1 + -2(2x) + \frac{-2.-3}{1.2}(2x)^2 + \frac{-2.-3.-4}{1.2.3}(2x)^3$

$= 1 - 4x + 12x^2 - 32x^3$

Valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

4iii) $\frac{1}{(1-x)(1+2x)^2} = (1-x)^{-1}(1+2x)^{-2}$

$(1-x)^{-1} \approx 1 + x + x^2 + x^3$

$(1+2x)^{-2} \approx 1 - 2(2x) + \frac{-2.-3}{1.2}(2x)^2 + \frac{-2.-3.-4}{1.2.3}(2x)^3$

$= 1 - 4x + 12x^2 - 32x^3$

$(1+x+x^2+x^3)(1-4x+12x^2-32x^3)$

$\approx 1 + x + x^2 + x^3$
$\quad -4x - 4x^2 - 4x^3$
$\quad +12x^2 + 12x^3$
$\quad -32x^3$

$= 1 - 3x + 9x^2 - 23x^3$

Valid for $|2x| < 1 \Rightarrow |x| < \frac{1}{2}$

5) i) $\frac{1}{\sqrt{4-x}} = \frac{1}{\sqrt{4(1-\frac{x}{4})}}$

$= \frac{1}{2\sqrt{1-\frac{x}{4}}} = \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$

ii) $\left(1-\frac{x}{4}\right)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\left(-\frac{x}{4}\right) + \frac{-\frac{1}{2}.-\frac{3}{2}}{1.2}\left(-\frac{x}{4}\right)^2$

$= 1 + \frac{x}{8} + \frac{3}{128}x^2$ Valid for $|x| < 4$

iii) $\frac{2(1+x)}{\sqrt{4-x}}$

$\approx 2(1+x)\frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$

$\approx (1+x)\left(1 + \frac{x}{8} + \frac{3}{128}x^2\right)$

$\approx 1 + x + \frac{x}{8} + \frac{x^2}{8} + \frac{3}{128}x^2$

$= 1 + \frac{9x}{8} + \frac{19}{128}x^2$

6) i) $(1+y)^{-1} \approx 1 - y + y^2 - y^3$

ii) $\left(1+\frac{2}{x}\right)^{-1} \approx 1 - \frac{2}{x} + \frac{4}{x^2} - \frac{8}{x^3}$

iii) $\left(1+\frac{2}{x}\right)^{-1} = \left(\frac{x+2}{x}\right)^{-1} = \frac{x}{x+2}$

$\frac{x}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{x}{2}\left(\frac{2+x}{2}\right)^{-1}$

$= \frac{x}{2}\left(\frac{2}{x+2}\right) = \frac{x}{x+2}$

iv) $\frac{x}{2}\left(1+\frac{x}{2}\right)^{-1}$

$\approx \frac{x}{2}\left[1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8}\right]$

$\approx \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{8} - \frac{x^4}{16}$

Valid for $\left|\frac{x}{2}\right| < 1 \Rightarrow |x| < 2$

6v) cont) $\left(1+\frac{2}{x}\right)^{-1}$ expansion valid

for $\left|\frac{2}{x}\right| < 1$

$\Rightarrow \quad |x| > 2$

No overlap in range of validity

7) i) $\displaystyle\int_1^k t \ln t \, dt$

Let $u = \ln t$ $\quad\Rightarrow \frac{du}{dt} = \frac{1}{t}$

Let $\frac{dv}{dt} = t$ $\quad\Rightarrow v = \frac{t^2}{2}$

$$\int u\frac{dv}{dt} = uv - \int v\frac{du}{dt}$$

$$\int_1^k t\ln t\,dt = \left[\frac{t^2}{2}\ln t\right]_1^k - \int_1^k \frac{t}{2}\,dt$$

$$= \left[\frac{k^2}{2}\ln k - 0\right] - \left[\frac{t^2}{4}\right]_1^k$$

$$= \frac{k^2}{2}\ln k - \frac{k^2}{4} + \frac{1}{4}$$

ii) $(1-2x)^{-\frac{1}{2}}$

$$\approx 1 - \tfrac{1}{2}(-2x) + \frac{-\frac{1}{2}\cdot-\frac{3}{2}}{1.2}(-2x)^2$$

$$+ \frac{-\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}}{1.2.3}(-2x)^3$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3$$ Valid for $|x| < \frac{1}{2}$

7iii) $(1-2x)^{-\frac{1}{2}}\ln(1+x)$

$$\approx \left(1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3\right)\ln(1+x)$$

$$\approx (1+x)\ln(1+x)$$

when x is small

$= t\ln t$ where $t = 1+x$

$$\therefore \int_0^{0.1} \frac{\ln(1+x)}{\sqrt{1-2x}}\,dx$$

$$= \int_0^{0.1} (1-2x)^{-\frac{1}{2}}\ln(1+x)\,dx$$

$$\approx \int_0^{0.1} (1+x)\ln(1+x)\,dx$$

$$= \int_1^{1.1} t\ln t\,dt$$

$$\approx \frac{1.1^2}{2}\ln 1.1 - \frac{1.1^2}{4} + \frac{1}{4}$$

$$\approx 0.00516$$