MEI CORE 4 TRIGONOMETRIC EQUATIONS EXERCISE 8C

1 i)

$$2\sin 2\theta = \cos\theta$$

$$4\sin\theta\cos\theta = \cos\theta$$

$$4\sin\theta\cos\theta - \cos\theta = 0$$

$$\cos\theta(4\sin\theta - 1) = 0$$

$$\Rightarrow \cos\theta = 0 \text{ or } 4\sin\theta - 1 = 0$$

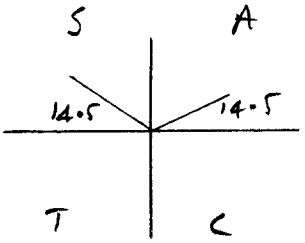
$$\cos\theta = 0 \Rightarrow \theta = 90^\circ, 270^\circ$$

$$4\sin\theta - 1 = 0$$

$$\Rightarrow 4\sin\theta = 1$$

$$\sin\theta = \frac{1}{4}$$

$$\sin^{-1}\frac{1}{4} = 14.5^\circ$$



$$\theta = 14.5^\circ, 165.5^\circ$$

Solution:

$$\theta = 14.5^\circ, 90^\circ, 165.5^\circ, 270^\circ$$

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1 ii)

$$\tan 2\theta = 4\tan\theta$$

$$\frac{2\tan\theta}{1-\tan^2\theta} = 4\tan\theta$$

$$2\tan\theta = 4\tan\theta(1-\tan^2\theta)$$

$$2\tan\theta = 4\tan\theta - 4\tan^3\theta$$

$$0 = 4\tan\theta - 4\tan^3\theta - 2\tan\theta$$

$$0 = 2\tan\theta - 4\tan^3\theta$$

$$0 = 2\tan\theta(1 - 2\tan^2\theta)$$

$$\Rightarrow \tan\theta = 0 \text{ or } 1 - 2\tan^2\theta = 0$$

$$\tan\theta = 0 \Rightarrow \theta = 0, 180^\circ, 360^\circ$$

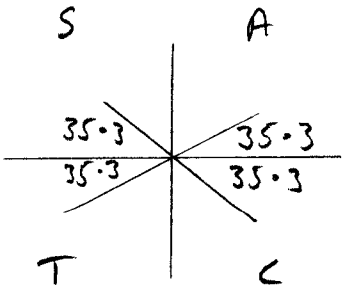
$$1 - 2\tan^2\theta = 0$$

$$\Rightarrow 1 = 2\tan^2\theta$$

$$\Rightarrow \frac{1}{2} = \tan^2\theta$$

$$\Rightarrow \tan\theta = \pm\frac{1}{\sqrt{2}}$$

$$\tan^{-1}\frac{1}{\sqrt{2}} = 35.3^\circ$$



$$\theta = 35.3^\circ, 144.7^\circ, 215.3^\circ, 324.7^\circ$$

Solution:

$$\theta = 0^\circ, 35.7^\circ, 144.7^\circ, 180^\circ$$
$$215.3^\circ, 324.7^\circ, 360^\circ$$

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1 iii) $\cos 2\theta + \sin\theta = 0$

$1 - 2\sin^2\theta + \sin\theta = 0$

$0 = 2\sin^2\theta - \sin\theta - 1$

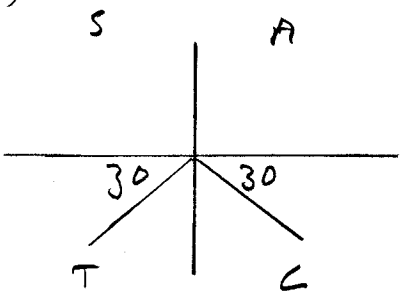
$0 = (2\sin\theta + 1)(\sin\theta - 1)$

Either $2\sin\theta + 1 = 0$

$2\sin\theta = -1$

$\sin\theta = -\frac{1}{2}$

$\sin^{-1}(\frac{1}{2}) = 30°$



$\theta = 210°, 330°$

or

$\sin\theta - 1 = 0$

$\sin\theta = 1$

$\theta = 90°$

Solution:

$\theta = 90°, 210°, 330°$

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1 iv) $\tan\theta \tan 2\theta = 1$

$\tan\theta \left(\frac{2\tan\theta}{1 - \tan^2\theta} \right) = 1$

$2\tan^2\theta = 1 - \tan^2\theta$

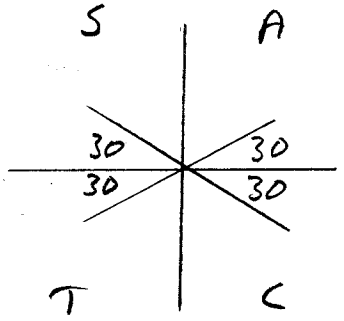
$2\tan^2\theta + \tan^2\theta = 1$

$3\tan^2\theta = 1$

$\tan^2\theta = \frac{1}{3}$

$\tan\theta = \pm\frac{1}{\sqrt{3}}$

$\tan^{-1}\frac{1}{\sqrt{3}} = 30°$



Solution:

$\theta = 30°, 150°, 210°, 330°$

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1v) $2\cos 2\theta = 1 + \cos\theta$

$2(2\cos^2\theta - 1) = 1 + \cos\theta$

$4\cos^2\theta - 2 = 1 + \cos\theta$

$4\cos^2\theta - \cos\theta - 3 = 0$

$(4\cos\theta + 3)(\cos\theta - 1) = 0$

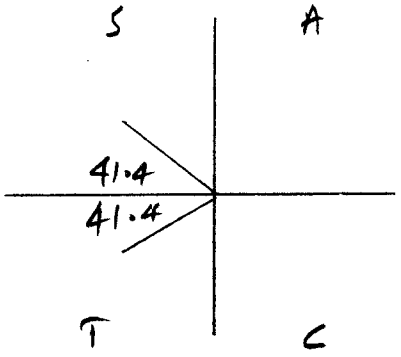
Either

$4\cos\theta + 3 = 0$

$4\cos\theta = -3$

$\cos\theta = -\frac{3}{4}$

$\cos^{-1}\frac{3}{4} = 41.4^\circ$



$\theta = 138.6^\circ, 221.4^\circ$

or $\cos\theta - 1 = 0$

$\cos\theta = 1$

$\theta = 0^\circ, 360^\circ$

Solution:

$\theta = 0^\circ, 138.6^\circ, 221.4^\circ, 360^\circ$

2i) $\sin 2\theta = 2\sin\theta$

$2\sin\theta\cos\theta = 2\sin\theta$

$2\sin\theta\cos\theta - 2\sin\theta = 0$

$2\sin\theta(\cos\theta - 1) = 0$

Either $\sin\theta = 0$

$\Rightarrow \theta = 0, \pi, -\pi$

or $\cos\theta - 1 = 0$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0$

Solution:

$\theta = -\pi, 0, \pi$

2ii) $\tan 2\theta = 2\tan\theta$

$\frac{2\tan\theta}{1 - \tan^2\theta} = 2\tan\theta$

$\frac{\tan\theta}{1 - \tan^2\theta} = \tan\theta$

$\tan\theta = \tan\theta(1 - \tan^2\theta)$

$\tan\theta = \tan\theta - \tan^3\theta$

$0 = \tan\theta - \tan^3\theta - \tan\theta$

$0 = -\tan^3\theta$

$\Rightarrow \tan^3\theta = 0$

$\Rightarrow \tan\theta = 0$

$\Rightarrow \theta = -\pi, 0, \pi$

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2iii) $\cos 2\theta - \cos\theta = 0$

$2\cos^2\theta - 1 - \cos\theta = 0$

$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

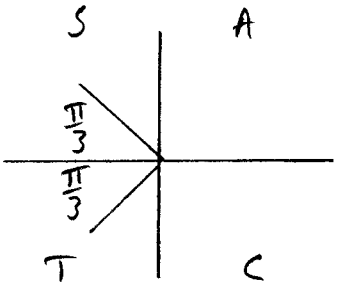
Either

$2\cos\theta + 1 = 0$

$\Rightarrow \quad 2\cos\theta = -1$

$\Rightarrow \quad \cos\theta = -\frac{1}{2}$

$\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$



$\theta = \frac{2\pi}{3}, -\frac{2\pi}{3}$

or $\cos\theta - 1 = 0$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0$

Solution:

$\theta = -\frac{2\pi}{3}, 0, \frac{2\pi}{3}$

2iv) $1 + \cos 2\theta = 2\sin^2\theta$

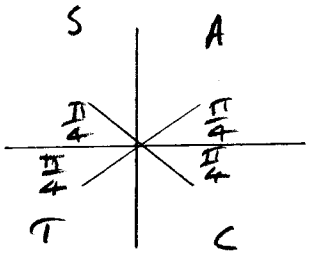
$1 + 1 - 2\sin^2\theta = 2\sin^2\theta$

$2 = 4\sin^2\theta$

$\frac{1}{2} = \sin^2\theta$

$\pm\frac{1}{\sqrt{2}} = \sin\theta$

$\sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$



$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{\pi}{4}, -\frac{3\pi}{4}$

Alternative Method

$1 + \cos 2\theta = 2\sin^2\theta$

$1 + 2\cos^2\theta - 1 = 2\sin^2\theta$

$2\cos^2\theta = 2\sin^2\theta$

$\cos^2\theta = \sin^2\theta$

$1 = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

$\Rightarrow \quad \tan\theta = \pm 1$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, -\frac{\pi}{4}, -\frac{3\pi}{4}$

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2v) $\sin 4\theta = \cos 2\theta$

$2\sin 2\theta \cos 2\theta = \cos 2\theta$

$2\sin 2\theta \cos 2\theta - \cos 2\theta = 0$

$\cos 2\theta (2\sin 2\theta - 1) = 0$

Either $\cos 2\theta = 0$

$\Rightarrow 2\theta = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

$\Rightarrow \theta = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

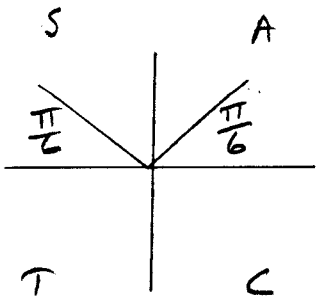
or

$2\sin 2\theta - 1 = 0$

$2\sin 2\theta = 1$

$\sin 2\theta = \frac{1}{2}$

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$



Going round CAST diagram both ways

$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$-\frac{7\pi}{6}, -\frac{11\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$

$-\frac{7\pi}{12}, -\frac{11\pi}{12}$

3) Write $\sin 3\theta$ in terms of $\sin\theta$

$\sin 3\theta = \sin(2\theta + \theta)$

$= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta$

$= 2\sin\theta\cos\theta\cos\theta + (1 - 2\sin^2\theta)\sin\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$

$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

Solve $\sin 3\theta = \sin\theta$

$3\sin\theta - 4\sin^3\theta = \sin\theta$

$0 = 4\sin^3\theta - 2\sin\theta$

$0 = 2\sin\theta(2\sin^2\theta - 1)$

Either

$\sin\theta = 0$

$\Rightarrow \theta = 0, \pi, 2\pi$

or

$2\sin^2\theta - 1 = 0$

$\Rightarrow 2\sin^2\theta = 1$

$\Rightarrow \sin^2\theta = \frac{1}{2}$

$\Rightarrow \sin\theta = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

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4) $\cos 3\theta = 1 - 3\cos\theta$

$\cos(2\theta + \theta) = 1 - 3\cos\theta$

$\cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= 1 - 3\cos\theta$

$(2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= 1 - 3\cos\theta$

$2\cos^3\theta - \cos\theta - 2\cos\theta(\sin^2\theta)$

$= 1 - 3\cos\theta$

$2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 1 - 3\cos\theta$

$2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 1 - 3\cos\theta$

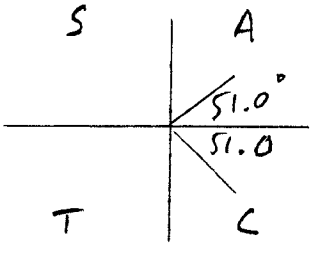
$4\cos^3\theta - 3\cos\theta = 1 - 3\cos\theta$

$4\cos^3\theta = 1$

$\cos^3\theta = \frac{1}{4}$

$\cos\theta = \sqrt[3]{\frac{1}{4}}$

$\cos^{-1}\left(\sqrt[3]{\frac{1}{4}}\right) = 51.0^\circ$



$\theta = 51.0^\circ, 309.0^\circ$

5) $\frac{1 + \cos 2\theta}{\sin 2\theta}$

$= \frac{1 + 2\cos^2\theta - 1}{2\sin\theta\cos\theta}$

$= \frac{2\cos^2\theta}{2\sin\theta\cos\theta}$

$= \frac{\cos\theta}{\sin\theta} = \cot\theta$

6) $\tan 3\theta = \tan(2\theta + \theta)$

$= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\tan\theta}$

$= \frac{\frac{2\tan\theta}{1 - \tan^2\theta} + \tan\theta}{1 - \tan\theta\left(\frac{2\tan\theta}{1 - \tan^2\theta}\right)}$

$= \frac{\frac{2\tan\theta + \tan\theta(1 - \tan^2\theta)}{1 - \tan^2\theta}}{\frac{1 - \tan^2\theta - 2\tan^2\theta}{1 - \tan^2\theta}}$

Multiply top and bottom by $1 - \tan^2\theta$

$= \frac{2\tan\theta + \tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

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7) $$\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$= \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

Multiply top and bottom by $\cos^2\theta$

$$= \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$

$$= \frac{\cos 2\theta}{1}$$

$$= \cos 2\theta$$

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8) i) $$\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{\pi}{4}-\theta\right)$$

$$= \left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right)\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)$$

$$= \left(\frac{1+\tan\theta}{1-\tan\theta}\right)\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= 1$$

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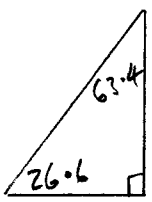
ii)



$$\tan 26.6 = \frac{1}{\tan 63.4}$$

If $\tan 26.6 = \frac{1}{2}$

$$\tan 63.4 = \frac{2}{1} = 2$$

$\therefore$ if $\tan\theta = 2 \quad 0 \le \theta \le 90$

$$\theta = 63.4^\circ$$

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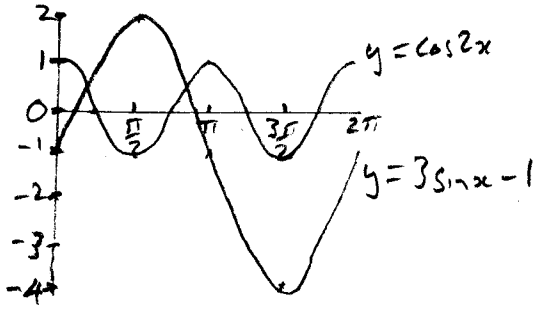
9)



$$y = \cos 2x$$

$$y = 3\sin x - 1$$

Meet when

$$\cos 2x = 3\sin x - 1$$

$$\Rightarrow 1 - 2\sin^2 x = 3\sin x - 1$$

$$0 = 2\sin^2 x + 3\sin x - 2$$

$$(2\sin x - 1)(\sin x + 2) = 0$$

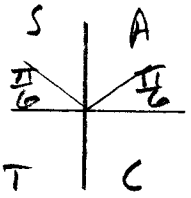
Either $2\sin x - 1 = 0$ or $\sin x + 2 = 0$

$\sin x = -2$ ✗

$$\Rightarrow 2\sin x = 1$$

$$\sin x = \frac{1}{2}$$

$\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$



Solution $x = \frac{\pi}{6}, \frac{5\pi}{6}$
