MEI FP1 MATRICES EXERCISE 1E

1) i) Let $\underline{A} = \begin{pmatrix} 4 & 3 \\ 6 & 5 \end{pmatrix}$ $\det \underline{A} = 2$

$A^{-1} = \frac{1}{2}\begin{pmatrix} 5 & -3 \\ -6 & 4 \end{pmatrix} = \begin{pmatrix} 5/2 & -\frac{3}{2} \\ -3 & 2 \end{pmatrix}$

ii) Let $A = \begin{pmatrix} 6 & -3 \\ -4 & -2 \end{pmatrix}$ $\det A = -24$

$A^{-1} = -\frac{1}{24}\begin{pmatrix} -2 & 3 \\ 4 & 6 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{12} & -\frac{1}{8} \\ -\frac{1}{6} & -\frac{1}{4} \end{pmatrix}$

iii) Let $A = \begin{pmatrix} 4 & 2 \\ -6 & -3 \end{pmatrix}$ $\det A = 0$

Singular $\therefore$ no inverse

iv) Let $A = \begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$ $\det A = 3$

$A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix}$

$= \begin{pmatrix} 1 & -2 \\ -\frac{2}{3} & 5/3 \end{pmatrix}$

v) Let $A = \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix}$ $\det A = -5$

$A^{-1} = -\frac{1}{5}\begin{pmatrix} -1 & -1 \\ -2 & 3 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{3}{5} \end{pmatrix}$

vi) Let $A = \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$ $\det A = 1$

$A^{-1} = \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$

vii) Let $A = \begin{pmatrix} 3 & -9 \\ -2 & 6 \end{pmatrix}$ $\det A = 0$

Singular $\therefore$ no inverse

viii) Let $A = \begin{pmatrix} \frac{1}{3} & \frac{3}{4} \\ \frac{2}{3} & 2 \end{pmatrix}$ $\det A = \frac{1}{6}$

$A^{-1} = 6\begin{pmatrix} 2 & -\frac{3}{4} \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

$= \begin{pmatrix} 12 & -\frac{9}{2} \\ -4 & 2 \end{pmatrix}$

ix) Let $A = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$\det A = eh - fg$

$A^{-1} = \frac{1}{eh-fg}\begin{pmatrix} h & -f \\ -g & e \end{pmatrix}$

provided that $eh - fg \neq 0$

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2) $\begin{pmatrix} 1-k & 2 \\ -1 & 4-k \end{pmatrix}$ is singular

$$\therefore (1-k)(4-k)+2=0$$
$$4-4k-k+k^2+2=0$$
$$k^2-5k+6=0$$
$$(k-3)(k-2)=0$$
$$\Rightarrow k=3 \text{ or } k=2$$

3) $A=\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}$ $B=\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$

i) $$A^{-1} = \frac{1}{2}\begin{pmatrix} 4 & -3 \\ -6 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -\frac{3}{2} \\ -3 & \frac{5}{2} \end{pmatrix}$$

ii) $$B^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

iii) $$A^{-1}B^{-1} = \begin{pmatrix} 2 & -\frac{3}{2} \\ -3 & \frac{5}{2} \end{pmatrix}\begin{pmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5}+\frac{3}{10} & -\frac{6}{5}-\frac{12}{10} \\ -\frac{6}{5}-\frac{1}{2} & \frac{9}{5}+2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{11}{10} & -\frac{24}{10} \\ -\frac{17}{10} & \frac{38}{10} \end{pmatrix}$$

iv) $$B^{-1}A^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{pmatrix}\begin{pmatrix} 2 & -\frac{3}{2} \\ -3 & \frac{5}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{5}+\frac{9}{5} & -\frac{3}{5}-\frac{15}{10} \\ -\frac{2}{5}-\frac{12}{5} & \frac{3}{10}+2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{26}{10} & -\frac{21}{10} \\ -\frac{28}{10} & \frac{23}{10} \end{pmatrix}$$

v) $$\underline{B}\,\underline{A} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 38 & 24 \\ 17 & 11 \end{pmatrix}$$

$$(BA)^{-1} = \frac{1}{38\times 11 - 17\times 24}\begin{pmatrix} 11 & -24 \\ -17 & 38 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 11 & -24 \\ -17 & 38 \end{pmatrix}$$

$$= \begin{pmatrix} 11 & -24 \\ -17 & 38 \end{pmatrix}$$

$$= A^{-1}B^{-1}$$

$$\therefore (BA)^{-1} = A^{-1}B^{-1}$$

3 cont) vi) $\underline{AB} = \begin{pmatrix} 5 & 3 \\ 6 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$

$\underline{AB} = \begin{pmatrix} 23 & 21 \\ 28 & 26 \end{pmatrix}$

$\det(\underline{AB}) = 23 \times 26 - 28 \times 21$
$= 10$

$(AB)^{-1} = \frac{1}{10}\begin{pmatrix} 26 & -21 \\ -28 & 23 \end{pmatrix}$

$= \begin{pmatrix} \frac{26}{10} & -\frac{21}{10} \\ -\frac{28}{10} & \frac{23}{10} \end{pmatrix}$

$= B^{-1}A^{-1}$

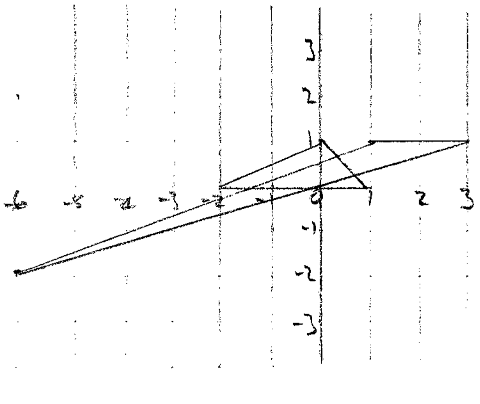
$\therefore (AB)^{-1} = B^{-1}A^{-1}$

4)

i) $\underline{T} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}$

$\underline{M}\,\underline{T} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}$

$\underline{T}' = \begin{pmatrix} 3 & 1 & -6 \\ 1 & 1 & -2 \end{pmatrix}$



ii) Area of $T = \frac{1}{2} \times 3 \times 1 = \frac{3}{2}$

Area of $T' = \frac{1}{2} \times 2 \times 3 = 3$

Ratio $\frac{\text{Area } T'}{\text{Area } T} = \frac{3}{3/2} = 2$

$\det M = 3 \times 1 - 1 \times 1 = 2$

iii) $\underline{M}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 3/2 \end{pmatrix}\begin{pmatrix} 3 & 1 & -6 \\ 1 & 1 & -2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \end{pmatrix}$

$\therefore \underline{M}^{-1}$ maps T' to T

5) $\underline{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$\underline{M}^2 = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$= \begin{pmatrix} a^2+bc & ac+cd \\ ab+bd & bc+d^2 \end{pmatrix}$

But since singular $ad = bc$

$= \begin{pmatrix} a^2+ad & ac+cd \\ ab+bd & ad+d^2 \end{pmatrix}$

5 cont)

$$= \begin{pmatrix} a(a+d) & c(a+d) \\ b(a+d) & d(a+d) \end{pmatrix}$$

$$= (a+d)\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$= (a+d)M$$

$$\therefore \quad M^n = (a+d)^{n-1}M$$

6) i)

$$(MN)(N^{-1}M^{-1})$$

$$= M(NN^{-1})M^{-1}$$

$$= MIM^{-1}$$

$$= MM^{-1}$$

$$= I$$

$$\therefore \quad (MN)^{-1} = N^{-1}M^{-1}$$

6 ii) $\underline{C} = \underline{A}\,\underline{B}$

a)

$$\underline{C} = \begin{pmatrix} 1 & 7 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix}$$

$$\underline{C} = \begin{pmatrix} 18 & -9 & 4 \\ 1 & -7 & 2 \\ -1 & -4 & 1 \end{pmatrix}$$

b)

$$\underline{A}\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 7 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a+7 & b+7c+4 \\ 0 & 1 & c+2 \\ 0 & 0 & 1 \end{pmatrix}$$

c) If

$$\begin{pmatrix} 1 & a+7 & b+7c+4 \\ 0 & 1 & c+2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow a = -7, \quad c = -2$$

$$b - 14 + 4 = 0 \quad \Rightarrow b = 10$$

$$\therefore \quad A^{-1} = \begin{pmatrix} 1 & -7 & 10 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

d)

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 3+a & 1 & 0 \\ -1-4a+b & -4+c & 1 \end{pmatrix}$$

If

$$\begin{pmatrix} 1 & 0 & 0 \\ 3+a & 1 & 0 \\ -1-4a+b & -4+c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow a = -3, \quad c = 4$$

$$-1 + 12 + b = 0 \quad \Rightarrow b = -11$$

$$\therefore \quad B^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & 4 & 0 \end{pmatrix}$$

6 cont) ii) e)

$$\underline{C} = \underline{A}\underline{B}$$

$$\underline{C}^{-1} = \underline{B}^{-1}\underline{A}^{-1}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -11 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & -7 & 10 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -7 & 10 \\ -3 & 22 & -32 \\ -11 & 81 & -117 \end{pmatrix}$$