# MEI FP1 MATRICES EXERCISE 1F

1) i) $\begin{vmatrix} 6 & 4 \\ 2 & 3 \end{vmatrix} = 18 - 8 = 10$

non-singular

ii) $\begin{vmatrix} 4 & 8 \\ -1 & -2 \end{vmatrix} = -8 - -8 = 0$

Singular

iii) $\begin{vmatrix} 5 & 3 \\ 1 & 3/5 \end{vmatrix} = 3 - 3 = 0$

Singular

iv) $\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} = 3 - -4 = 7$

non-singular

2) $\underline{M} = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$ $\underline{N} = \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$

i) $\begin{vmatrix} 5 & 3 \\ 4 & 2 \end{vmatrix} = 10 - 12 = -2$

$\det \underline{M} = -2$

$\begin{vmatrix} 3 & 2 \\ -2 & 1 \end{vmatrix} = 3 - -4 = 7$

$\det \underline{N} = 7$

ii) $\underline{M}\underline{N} = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 13 \\ 8 & 10 \end{pmatrix}$

$\det(\underline{M}\underline{N}) = 9 \times 10 - 8 \times 13$
$= 90 - 104$

$= -14$

$= \det \underline{M} \times \det \underline{N}$

iii) Premultiplying by $\underline{N}$ increases area by scale factor $\det \underline{N}$
Premultiplying by $\underline{M}$ then increases area by scale factor $\det \underline{M}$

Premultiplying by $\underline{M}\underline{N}$ has this effect so area increases by scale factor $\det \underline{M} \times \det \underline{N}$

$\therefore$ $\det(\underline{M}\underline{N})$ is this combined scale factor
$\therefore$ $\det(\underline{M}\underline{N}) = \det \underline{M} \times \det \underline{N}$

3) $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ preserves area

$\therefore$ $\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = 1$

$ad = 1$

4) $\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x + ky \\ y \end{pmatrix}$

Matrix is $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

$\det \begin{vmatrix} 1 & k \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

for all $k$

$\therefore$ the shear preserves area

5) i) $\underline{M} = \begin{pmatrix} 7 & -3 \\ -4 & 6 \end{pmatrix}$

$\det \underline{M} = 42 - 12 = 30$

5 cont) If S has area 5 units$^2$

T will have area $5 \times 30$
$= 150$ units$^2$

ii) Find inverse of $\begin{pmatrix} 7 & -3 \\ -4 & 6 \end{pmatrix}$

$$= \frac{1}{30}\begin{pmatrix} 6 & 3 \\ 4 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ \frac{2}{15} & \frac{7}{30} \end{pmatrix}$$

iii) Rotation $135^\circ$ anticlockwise about $(0,0)$

Transformation matrix is

$$\begin{pmatrix} \cos 135^\circ & -\sin 135^\circ \\ \sin 135^\circ & \cos 135^\circ \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

iv)

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ \frac{2}{15} & \frac{7}{30} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ \frac{2}{15} & \frac{7}{30} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{1}{5}-\frac{2}{15} & -\frac{1}{10}-\frac{7}{30} \\ \frac{1}{5}-\frac{2}{15} & \frac{1}{10}-\frac{7}{30} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{15} & -\frac{2}{15} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{1}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{15\sqrt{2}} & -\frac{2}{15\sqrt{2}} \end{pmatrix}$$

6)

$$\underline{M} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

i) $\det \underline{M} = 4 - 4 = 0$

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+4y \\ x+2y \end{pmatrix}$$

$$= \begin{pmatrix} 2k \\ k \end{pmatrix}$$

where $x+2y=k$

$\therefore$ plane mapped onto $y = \frac{1}{2}x$

or $2y = x$
or $x - 2y = 0$

ii) $P(x,y) \Rightarrow P'(4,2)$

$$\underline{p}' = \underline{M}\,\underline{p}$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \quad 2x + 4y = 4$$
$$x + 2y = 2$$

Satisfied by any point on $x + 2y = 2$

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6 cont) iii)

$$\begin{pmatrix}10\\5\end{pmatrix}=\begin{pmatrix}2&4\\1&2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\Rightarrow \begin{aligned}2x+4y&=10\\x+2y&=5\end{aligned}$$

Line is $x+2y=5$

7) i)

$$\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}1&2\\3&6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}x+2y\\3x+6y\end{pmatrix}$$

$$\underline{T}=\begin{pmatrix}1&2\\3&6\end{pmatrix}$$

$$\det \underline{T} = 6-6=0$$

ii)

$$\begin{pmatrix}1&2\\3&6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x+2y\\3x+6y\end{pmatrix}$$

$$=\begin{pmatrix}k\\3k\end{pmatrix}$$

where $k=x+2y$

$(k, 3k)$ is a point on $y=3x$

iii)

$$x+2y=3$$

Let point be $\left(x, \frac{3-x}{2}\right)$

$$\begin{pmatrix}1&2\\3&6\end{pmatrix}\begin{pmatrix}x\\\frac{3-x}{2}\end{pmatrix}$$

$$=\begin{pmatrix}x+3-x\\3x+9-3x\end{pmatrix}=\begin{pmatrix}3\\9\end{pmatrix}$$

Point is $(3, 9)$

8) i)

Gradient $=3$

$$y-y_1=m(x-x_1)$$

$$y-t=3(x-s)$$

$$\underline{y=3x-3s+t}$$

ii)

$$\begin{aligned}y&=3x-3s+t\\3y&=x\end{aligned}$$

$$\Rightarrow y=\frac{x}{3}$$

$$\frac{x}{3}=3x-3s+t$$

$$x=9x-9s+3t$$

$$9s-3t=8x$$

$$\frac{9s-3t}{8}=x$$

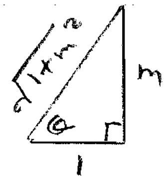
$$\frac{3s-t}{8}=y$$

$$P'=\left(\frac{9s-3t}{8}, \frac{3s-t}{8}\right)$$

$$\begin{pmatrix}9/8&-\frac{3}{8}\\3/8&-\frac{1}{8}\end{pmatrix}\begin{pmatrix}s\\t\end{pmatrix}=\begin{pmatrix}\frac{9s-3t}{8}\\\frac{3s-t}{8}\end{pmatrix}$$

$$\begin{vmatrix}9/8&-3/8\\3/8&-\frac{1}{8}\end{vmatrix}=-\frac{9}{64}+\frac{9}{64}=0$$

9) i) $y = mx$




Rotate $\theta^\circ$ clockwise, apply shear parallel to $x$ axis, then rotate $\theta^\circ$ anticlockwise

Transformation becomes

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{\sqrt{1+m^2}} & \frac{-m}{\sqrt{1+m^2}} \\ \frac{m}{\sqrt{1+m^2}} & \frac{1}{\sqrt{1+m^2}} \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1+m^2}} & \frac{m}{\sqrt{1+m^2}} \\ \frac{-m}{\sqrt{1+m^2}} & \frac{1}{\sqrt{1+m^2}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{(1+m^2)}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Image of $I(1,0)$

$$\frac{1}{(1+m^2)}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{1+m^2}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -m \end{pmatrix}$$

$$= \frac{1}{(1+m^2)}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1+mk \\ -m \end{pmatrix}$$

$$= \frac{1}{(1+m^2)}\begin{pmatrix} 1+mk+m^2 \\ m+m^2k-m \end{pmatrix}$$

$$= \frac{1}{(1+m^2)}\begin{pmatrix} 1+mk+m^2 \\ m^2k \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1+mk+m^2}{1+m^2} \\ \frac{m^2k}{1+m^2} \end{pmatrix}$$

Image of $J(0,1)$

$$\frac{1}{(1+m^2)}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & m \\ -m & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{(1+m^2)}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} m \\ 1 \end{pmatrix}$$

$$= \frac{1}{(1+m^2)}\begin{pmatrix} 1 & -m \\ m & 1 \end{pmatrix}\begin{pmatrix} m-k \\ 1 \end{pmatrix}$$

$$= \frac{1}{(1+m^2)}\begin{pmatrix} m-k-m \\ m^2-km+1 \end{pmatrix}$$

$$= \frac{1}{(1+m^2)}\begin{pmatrix} -k \\ m^2-km+1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-k}{1+m^2} \\ \frac{m^2-km+1}{1+m^2} \end{pmatrix}$$

Transformation matrix is

ii)

$$\begin{pmatrix} \frac{1+mk+m^2}{1+m^2} & \frac{-k}{1+m^2} \\ \frac{m^2k}{1+m^2} & \frac{m^2-km+1}{1+m^2} \end{pmatrix}$$

9 cont)
iii)

$$\begin{vmatrix} \frac{1+mk+m^2}{1+m^2} & \frac{-k}{1+m^2} \\ \frac{m^2k}{1+m^2} & \frac{m^2-km+1}{1+m^2} \end{vmatrix}$$

$$= \frac{1}{(1+m^2)^2}\left[(1+mk+m^2)(m^2-km+1) + k^2m^2\right]$$

$$= \frac{1}{(1+m^2)^2}\left(m^2 + m^3k + m^4 \cancel{-km} \cancel{-k^2m^2} - km^3 + 1 \cancel{+mk} + m^2 \cancel{+k^2m^2}\right)$$

$$= \frac{1}{(1+m^2)^2}\left(m^4 + 2m^2 + 1\right)$$

$$= \frac{(1+m^2)^2}{(1+m^2)^2} = 1$$

$\therefore$ area preserved whatever the value of m