1) Let $C = 1 + \cos\theta + \cos 2\theta + \ldots + \cos(n-1)\theta$
and $S = \sin\theta + \sin 2\theta + \ldots + \sin(n-1)\theta$

Then $C + jS = 1 + (\cos\theta + j\sin\theta) + (\cos 2\theta + j\sin 2\theta) + \ldots + (\cos(n-1)\theta + j\sin(n-1)\theta)$

$= 1 + e^{j\theta} + e^{j2\theta} + \ldots + e^{j(n-1)\theta}$

This is a GP with $a = 1$, $r = e^{j\theta}$

$$\text{Sum} = \frac{a(1-r^n)}{1-r} = \frac{1(1-e^{jn\theta})}{1-e^{j\theta}}$$

$$C + jS = \frac{(1-e^{jn\theta})}{(1-e^{j\theta})} \times \frac{(1-e^{-j\theta})}{(1-e^{-j\theta})}$$

$$C + jS = \frac{1 - e^{jn\theta} - e^{-j\theta} + e^{j(n-1)\theta}}{1 - e^{j\theta} - e^{-j\theta} + 1}$$

$$C + jS = \frac{1 - (\cos n\theta + j\sin n\theta) - (\cos\theta - j\sin\theta) + (\cos(n-1)\theta + j\sin(n-1)\theta)}{2 - (\cos\theta + j\sin\theta) - (\cos\theta - j\sin\theta)}$$

$$C + jS = \frac{1 - \cos n\theta - \cos\theta + \cos(n-1)\theta + j(-\sin n\theta + \sin\theta + \sin(n-1)\theta)}{2 - 2\cos\theta}$$

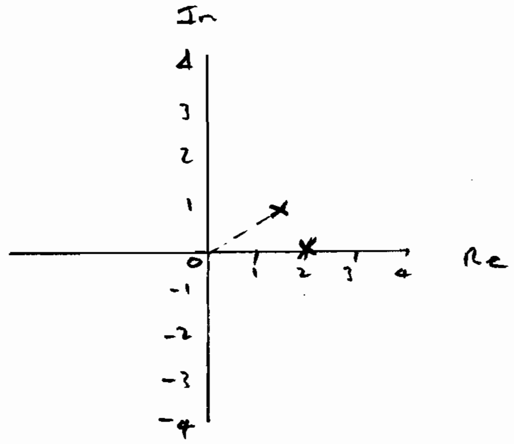
Equating real and imaginary parts

$$C = \frac{1 - \cos n\theta - \cos\theta + \cos(n-1)\theta}{2 - 2\cos\theta}$$

$$S = \frac{\sin\theta + \sin(n-1)\theta - \sin n\theta}{2 - 2\cos\theta}$$

2)

i)



Let $A = 2 + \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}$

$= 2 - \frac{1}{2} + \frac{\sqrt{3}}{2}j$

$= \frac{3}{2} + \frac{\sqrt{3}}{2}j$

$|A| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$|A| = \sqrt{\frac{9}{4} + \frac{3}{4}} = \sqrt{\frac{12}{4}} = \sqrt{3}$

Find $\operatorname{Arg} A = \alpha$ say

$\alpha = \tan^{-1} \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \tan^{-1}\left(\frac{\sqrt{3}}{2} \times \frac{2}{3}\right)$

$\alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

$\therefore \quad A = \sqrt{3}\, e^{j\frac{\pi}{6}}$

2ii) $\sum_{r=0}^{n} {}^nC_r\, 2^{n-r} \cos\frac{2r\pi}{3}$

$= 2^n \cos 0 + {}^nC_1\, 2^{n-1}\cos\frac{2\pi}{3} + {}^nC_2\, 2^{n-2}\cos\frac{4\pi}{3} + \dots + {}^nC_n\, 2^0 \cos\frac{2n\pi}{3} \quad (*)$

Consider $\left(2 + \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right)^n$

$= 2^n + {}^nC_1\, 2^{n-1}\left(\cos\frac{2\pi}{3} + j\sin\frac{\pi}{3}\right) + {}^nC_2\, 2^{n-2}\left(\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right)^2 + \dots$

$+ {}^nC_n\left(\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right)^n$

This is the cosine series given together with a similar sine series.

Now $\left(2 + \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}\right)^n = \left(\sqrt{3}\, e^{j\frac{\pi}{6}}\right)^n = 3^{\frac{n}{2}} e^{j\frac{n\pi}{6}}$

$= 3^{\frac{n}{2}}\left(\cos\frac{n\pi}{6} + j\sin\frac{n\pi}{6}\right)$

Equating real and imaginary parts

Series $(*) = 3^{\frac{n}{2}} \cos\frac{n\pi}{6}$

2 iii) Sine result

$$\sum_{r=0}^{n} {}^nC_r \sin\frac{2r\pi}{3} = 3^{\frac{n}{2}} \sin\frac{n\pi}{6}$$

3 i)

$$w = 1 - e^{j\theta}\cos\theta \qquad e^{jk\theta} = \cos k\theta + j\sin k\theta$$
$$e^{-jk\theta} = \cos k\theta - j\sin k\theta$$

$$w = 1 - (\cos\theta + j\sin\theta)\cos\theta$$
$$w = 1 - \cos^2\theta - j\sin\theta\cos\theta$$
$$w = \sin^2\theta - j\sin\theta\cos\theta$$
$$w = \sin\theta(\sin\theta - j\cos\theta) \qquad \text{multiply by } (-j)(+j) = 1$$
$$w = -j\sin\theta(+j\sin\theta - j^2\cos\theta)$$
$$w = -j\sin\theta(\cos\theta + j\sin\theta) = -j\sin\theta\, e^{j\theta}$$

3 ii)

$$|w| = |\sin^2\theta - j\sin\theta\cos\theta|$$
$$= \sqrt{\sin^4\theta + \sin^2\theta\cos^2\theta}$$
$$= \sqrt{\sin^2\theta(\sin^2\theta + \cos^2\theta)}$$
$$= \sin\theta\sqrt{1} = \sin\theta$$

$$\arg w = \theta - \frac{\pi}{2}$$

because $\arg(e^{j\theta}) = \theta$

$\arg -j\sin\theta = -\frac{\pi}{2}$

$\therefore \arg(e^{j\theta} \times (-j\sin\theta)) = \theta + (-\frac{\pi}{2})$

$= \theta - \frac{\pi}{2}$

$$|\sqrt{w}| = \sqrt{\sin\theta} \qquad \arg\sqrt{w} = \frac{1}{2}\left(\theta - \frac{\pi}{2}\right) = \frac{\theta}{2} - \frac{\pi}{4}$$

$$\text{or } \frac{1}{2}\left(\theta + 2\pi - \frac{\pi}{2}\right) = \frac{\theta}{2} + \frac{3\pi}{4}$$

3 iii) $C = \cos\theta\cos\theta + \cos 2\theta\cos^2\theta + \cos 3\theta\cos^3\theta + + \cos n\theta\cos^n\theta$

$S = \sin\theta\cos\theta + \sin 2\theta\cos^2\theta + \sin 3\theta\cos^3\theta + + \sin n\theta\cos^n\theta$

$$C + jS = (\cos\theta + j\sin\theta)\cos\theta + (\cos 2\theta + j\sin 2\theta)\cos^2\theta + + (\cos n\theta + j\sin n\theta)\cos^n\theta$$

$$= e^{j\theta}\cos\theta + e^{j2\theta}\cos^2\theta + \dots + e^{jn\theta}\cos^n\theta$$

GP $a = e^{j\theta}\cos\theta$, $r = e^{j\theta}\cos\theta$

$$\text{Sum} = \frac{a(1-r^n)}{1-r} = \frac{e^{j\theta}\cos\theta(1 - e^{jn\theta}\cos^n\theta)}{1 - e^{j\theta}\cos\theta}$$

---

iv)

$$C + jS = \frac{e^{j\theta}\cos\theta(1 - e^{jn\theta}\cos^n\theta)}{1 - e^{j\theta}\cos\theta} = \frac{e^{j\theta}\cos\theta(1 - e^{jn\theta}\cos^n\theta)}{-je^{j\theta}\sin\theta}$$

(using result from (i))

$$C + jS = j\frac{\cos\theta(1 - e^{jn\theta}\cos^n\theta)}{\sin\theta}$$

$$C + jS = j\frac{\cos\theta(1 - (\cos n\theta + j\sin n\theta)\cos^n\theta)}{\sin\theta}$$

$$C + jS = j\frac{\cos\theta(1 - \cos n\theta\cos^n\theta - j\sin n\theta\cos^n\theta)}{\sin\theta}$$

$$C + jS = \frac{j\cos\theta - j\cos n\theta\cos^{n+1}\theta + \sin n\theta\cos^{n+1}\theta}{\sin\theta}$$

Equating real and imaginary parts

$$C = \frac{\sin n\theta\cos^{n+1}\theta}{\sin\theta} \qquad S = \frac{\cos\theta - \cos n\theta\cos^{n+1}\theta}{\sin\theta}$$

4 i) $z = \cos\theta + j\sin\theta$ $\qquad z^3 = \cos 3\theta + j\sin 3\theta$

$z^2 = \cos 2\theta + j\sin 2\theta$ $\qquad z^n = \cos n\theta + j\sin n\theta$

ii)
$$C = 1 + \tfrac{1}{3}\cos\theta + \tfrac{1}{9}\cos 2\theta + \dots + \tfrac{1}{3^n}\cos n\theta$$
$$S = +\tfrac{1}{3}\sin\theta + \tfrac{1}{9}\sin 2\theta + \dots + \tfrac{1}{3^n}\sin n\theta$$
$$C + jS = 1 + \tfrac{1}{3}(\cos\theta + j\sin\theta) + \tfrac{1}{9}(\cos 2\theta + j\sin 2\theta) + \dots + \tfrac{1}{3^n}(\cos n\theta + j\sin n\theta)$$
$$= 1 + \tfrac{1}{3}z + \tfrac{1}{9}z^2 + \dots + \tfrac{1}{3^n}z^n$$

GP $a = 1$, $r = \tfrac{1}{3}z$

iii)
$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}z} = \frac{3}{3-z}$$

iv)
$$C + jS = \frac{3}{3 - \cos\theta - j\sin\theta} = \frac{3}{3 - \cos\theta - j\sin\theta} \times \frac{(3 - \cos\theta + j\sin\theta)}{(3 - \cos\theta + j\sin\theta)}$$
$$C + jS = \frac{9 - 3\cos\theta + 3j\sin\theta}{(3-\cos\theta)^2 + \sin^2\theta} = \frac{9 - 3\cos\theta + 3j\sin\theta}{9 - 6\cos\theta + \cos^2\theta + \sin^2\theta}$$
$$C + jS = \frac{9 - 3\cos\theta + 3j\sin\theta}{10 - 6\cos\theta}$$

Equating real and imaginary parts

$$C = \frac{9 - 3\cos\theta}{10 - 6\cos\theta} \qquad S = \frac{3\sin\theta}{10 - 6\cos\theta}$$

v)
$$|C + jS| = \sqrt{\left(\frac{9 - 3\cos\theta}{10 - 6\cos\theta}\right)^2 + \left(\frac{3\sin\theta}{10 - 6\cos\theta}\right)^2}$$
$$|C + jS| = \frac{\sqrt{81 - 54\cos\theta + 9\cos^2\theta + 9\sin^2\theta}}{10 - 6\cos\theta} = \frac{\sqrt{90 - 54\cos\theta}}{10 - 6\cos\theta}$$
$$|C + jS| = \frac{3\sqrt{10 - 6\cos\theta}}{10 - 6\cos\theta} = \frac{3}{\sqrt{10 - 6\cos\theta}}$$

5) i) $e^{j\theta} = \cos\theta + j\sin\theta$, $e^{jn\theta} = \cos n\theta + j\sin n\theta$, $e^{-jn\theta} = \cos n\theta - j\sin n\theta$

ii) $\left(1 - \frac{1}{2}e^{2j\theta}\right)\left(1 - \frac{1}{2}e^{-2j\theta}\right) = 1 - \frac{1}{2}e^{2j\theta} - \frac{1}{2}e^{-2j\theta} + \frac{1}{4}$

$= \frac{5}{4} - \frac{1}{2}\left(e^{2j\theta} + e^{-2j\theta}\right) = \frac{5}{4} - \frac{1}{2}\left(\cos 2\theta + j\sin 2\theta + \cos 2\theta - j\sin 2\theta\right)$

$= \frac{5}{4} - \frac{1}{2}\left(2\cos 2\theta\right) = \frac{5}{4} - \cos 2\theta$

iii) $C = \cos\theta + \frac{1}{2}\cos 3\theta + \frac{1}{4}\cos 5\theta + \frac{1}{8}\cos 7\theta + \ldots + \frac{1}{2^{r-1}}\cos(2r-1)\theta$

$S = \sin\theta + \frac{1}{2}\sin 3\theta + \frac{1}{4}\sin 5\theta + \frac{1}{8}\sin 7\theta + \ldots + \frac{1}{2^{r-1}}\sin(2r-1)\theta$

$C + jS = (\cos\theta + j\sin\theta) + \frac{1}{2}(\cos 3\theta + j\sin 3\theta) + \ldots + \frac{1}{2^{r-1}}(\cos(2r-1)\theta + j\sin(2r-1)\theta)$

$C + jS = e^{j\theta} + \frac{1}{2}e^{j3\theta} + \frac{1}{4}e^{j5\theta} + \ldots + \frac{1}{2^{r-1}}e^{j(2r-1)\theta}$

GP $a = e^{j\theta}$, $r = \frac{1}{2}e^{j2\theta}$

$$S_\infty = \frac{a}{1-r} = \frac{e^{j\theta}}{1 - \frac{1}{2}e^{j2\theta}} = \frac{2e^{j\theta}}{2 - e^{j2\theta}}$$

$$S_\infty = \frac{2e^{j\theta}}{2 - \cos 2\theta - j\sin 2\theta} = \frac{2e^{j\theta}}{2 - \cos 2\theta - j\sin 2\theta} \times \frac{2 - \cos 2\theta + j\sin 2\theta}{2 - \cos 2\theta + j\sin 2\theta}$$

$$S_\infty = \frac{2e^{j\theta}(2 - \cos 2\theta + j\sin 2\theta)}{(2 - \cos 2\theta)^2 + \sin^2 2\theta} = \frac{2e^{j\theta}(2 - e^{-2j\theta})}{4 - 4\cos 2\theta + 1}$$

$$S_\infty = \frac{4e^{j\theta} - 2e^{-j\theta}}{5 + 4\cos 2\theta}$$

5iv) $$C + jS = \frac{4(\cos\theta + j\sin\theta) - 2(\cos\theta - j\sin\theta)}{5 + 4\cos 2\theta}$$

Equating real and imaginary parts

$$C = \frac{4\cos\theta - 2\cos\theta}{5 + 4\cos 2\theta} = \frac{2\cos\theta}{5 + 4\cos 2\theta}$$

$$S = \frac{4\sin\theta + 2\sin\theta}{5 + 4\cos 2\theta} = \frac{6\sin\theta}{5 + 4\cos 2\theta}$$

---

6)

i) $$e^{jn\theta} = \cos n\theta + j\sin n\theta$$

ii) $$\tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) = \tfrac{1}{2}\left[\cos\theta + j\sin\theta + \cos\theta - j\sin\theta\right]$$
$$= \cos\theta$$

$$\left(1 + \tfrac{1}{2}e^{j\theta}\right)\left(1 + \tfrac{1}{2}e^{-j\theta}\right) = 1 + \tfrac{1}{2}e^{j\theta} + \tfrac{1}{2}e^{-j\theta} + \tfrac{1}{4}$$
$$= \frac{5}{4} + \cos\theta$$

iii)

$$C = \frac{\cos\theta}{2} - \frac{\cos 2\theta}{4} + \frac{\cos 3\theta}{8} - \frac{\cos 4\theta}{16} + \dots$$

$$S = \frac{\sin\theta}{2} - \frac{\sin 2\theta}{4} + \frac{\sin 3\theta}{8} - \frac{\sin 4\theta}{16} + \dots$$

$$C + jS = \tfrac{1}{2}(\cos\theta + j\sin\theta) - \tfrac{1}{4}(\cos 2\theta + j\sin 2\theta)$$
$$+ \tfrac{1}{8}(\cos 3\theta + j\sin 3\theta) - \tfrac{1}{16}(\cos 4\theta + j\sin 4\theta) + \dots$$

$$C + jS = \tfrac{1}{2}e^{j\theta} - \tfrac{1}{4}e^{2j\theta} + \tfrac{1}{8}e^{3j\theta} - \tfrac{1}{16}e^{4j\theta} + \dots$$

GP $a = \tfrac{1}{2}e^{j\theta}$, $r = -\tfrac{1}{2}e^{j\theta}$

$$C + jS = S_\infty = \frac{a}{1-r} = \frac{\tfrac{1}{2}e^{j\theta}}{1 + \tfrac{1}{2}e^{j\theta}} = \frac{e^{j\theta}}{2 + e^{j\theta}}$$

6 iii cont

$$C + jS = \frac{\cos\theta + j\sin\theta}{2+\cos\theta + j\sin\theta} \times \frac{(2+\cos\theta - j\sin\theta)}{(2+\cos\theta - j\sin\theta)}$$

$$= \frac{2\cos\theta + 2j\sin\theta + \cos^2\theta + \cancel{j\sin\theta\cos\theta} - \cancel{j\sin\theta\cos\theta} + \sin^2\theta}{(2+\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{2(\cos\theta + j\sin\theta) + 1}{4 + 4\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{2e^{j\theta} + 1}{5 + 4\cos\theta}$$

6iv)

$$C + jS = \frac{2\cos\theta + j\sin\theta + 1}{5 + 4\cos\theta}$$

Equating real and imaginary parts

$$C = \frac{2\cos\theta + 1}{5 + 4\cos\theta} \qquad S = \frac{\sin\theta}{5+4\cos\theta}$$

7)

Find $\sum_{r=0}^{n} {}^nC_r \sin(\alpha + r\beta) = S$ say

Let $C = \sum_{r=0}^{n} {}^nC_r \cos(\alpha + r\beta)$

$$C + jS = \sum_{r=0}^{n} {}^nC_r \left(\cos(\alpha + r\beta) + j\sin(\alpha + r\beta)\right)$$

$$= \sum_{r=0}^{n} {}^nC_r \, e^{(\alpha + r\beta)j}$$

$$= \sum_{r=0}^{n} {}^nC_r \, e^{\alpha j} e^{r\beta j}$$

$$= e^{\alpha j}(1 + e^{\beta j})^n$$

$$= e^{\alpha j}\left(2\cos\frac{\beta}{2} e^{\frac{\beta j}{2}}\right)^n$$

7 (cont)

$$C + jS = e^{\alpha j}\left(2^n \cos^n \frac{\beta}{2} e^{\frac{n\beta j}{2}}\right)$$

$$= 2^n \cos^n \frac{\beta}{2}\left(\cos\frac{\beta n}{2} + j\sin\frac{\beta n}{2}\right)\left(\cos\alpha + j\sin\alpha\right)$$

Equating imaginary parts

$$S = 2^n \cos^n \frac{\beta}{2}\left(\sin\alpha\cos\frac{\beta n}{2} + \cos\alpha\sin\frac{\beta n}{2}\right)$$

$$S = 2^n \cos^n \frac{\beta}{2} \sin\left(\alpha + \frac{\beta n}{2}\right)$$