1) Let $f(x) = \sin x$

i)

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$$f(x) \approx f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!}$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0$$

$\therefore$ for small x

$$\sin x \approx x - \frac{x^3}{6}$$

ii)

Let $f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f^{(3)}(x) = \sin x$

$f^{(4)}(x) = \cos x$

$$f(x) \approx f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + f^{(4)}(0)\frac{x^4}{4!}$$

$$= 1 + 0 - \frac{x^2}{2} + 0 + \frac{x^4}{4!}$$

for small x

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

iii)

Let $f(x) = \tan x$

$$f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$$

$$f''(x) = \frac{d}{dx}(\cos x)^{-2}$$

$$= -2(\cos x)^{-3} \times (-\sin x)$$

$$f''(x) = 2\sin x \cos^{-3} x$$

$$f'''(x) = 2\sin x \times (-3)\cos^{-4} x\,(-\sin x) + \cos^{-3}(x) \times 2\cos x$$

$$= +6\sin^2 x \cos^{-4} x + 2\cos^{-2} x$$

$$= 6(1-\cos^2 x)\cos^{-4} x + 2\cos^{-2} x$$

$$= 6\cos^{-4} x - 6\cos^{-2} x + 2\cos^{-2} x$$

$$= 6\cos^{-4} x - 4\cos^{-2} x$$

$$f^{(4)}(x) = -24\cos^{-5} x(-\sin x) + 8\cos^{-3} x(-\sin x)$$

$$= 8\sin x\left(3\cos^{-5} x - \cos^{-3} x\right)$$

$\Rightarrow f(0) = 0,\quad f'(0) = 1$

$f''(0) = 0,\quad f^{(3)}(0) = 6 - 4 = 2$

$f^{(4)}(0) = 0$

$\therefore$ for small angles

$$\tan x \approx x + 0 + \frac{2x^3}{6} + 0$$

$$= x + \frac{x^3}{3}$$

2) Spreadsheet question – omitted

3) Calculate $\frac{1}{\sqrt{e}}$ to 5 dp

Let $f(x) = e^{-\frac{1}{2}x}$

$f'(x) = -\frac{1}{2}e^{-\frac{1}{2}x}$

$f''(x) = \frac{1}{4}e^{-\frac{1}{2}x}$

$f'''(x) = -\frac{1}{8}e^{-\frac{1}{2}x}$

$f^{(4)}(x) = \frac{1}{16}e^{-\frac{1}{2}x}$

$\Rightarrow f(0) = 1, \quad f'(0) = -\frac{1}{2}$

$f''(0) = \frac{1}{4}, \quad f'''(0) = -\frac{1}{8} \quad f^{(4)}(0) = \frac{1}{16}$

$f(x) \approx 1 - \frac{x}{2} + \frac{1}{4}\cdot\frac{x^2}{2} - \frac{1}{8}\cdot\frac{x^3}{3!} + \frac{1}{16}\cdot\frac{x^4}{4!}$

$\approx 1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} + \frac{x^4}{384}$

Continuing pattern

$-\frac{1}{32}\cdot\frac{x^5}{5!} + \frac{1}{64}\frac{x^6}{6!} - \frac{1}{128}\cdot\frac{x^7}{7!}$

when $x = 1$, $f(1) = e^{-\frac{1}{2}}$

$\therefore \frac{1}{\sqrt{e}} \approx 1 - \frac{1}{2} + \frac{1}{8} - \frac{1}{48} + \frac{1}{384}$

$- \frac{1}{3840} + \frac{1}{46080} - \frac{1}{645120}$

≈ 0.60653 to 5 dp

4) $\sin x \approx x - \frac{x^3}{6}$

Solve $\sin x = x^2$

Solve $x - \frac{x^3}{6} = x^2$

$6x - x^3 = 6x^2$

$x^3 + 6x^2 - 6x = 0$

$x(x^2 + 6x - 6) = 0$

$x = 0$ or

$x = \frac{-6 \pm \sqrt{36+24}}{2}$

$x = \frac{-6 \pm 2\sqrt{15}}{2}$

$x = -3 \pm \sqrt{15}$

$x = 0, \quad x = -3 - \sqrt{15}, \quad x = \sqrt{15} - 3$

$-3 - \sqrt{15}$ too large for valid approximation

$\therefore x = 0$ or $x \approx \sqrt{15} - 3$

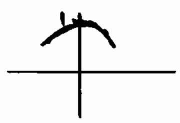
5) $f(x) = 1 - \frac{3}{2}x^2 + \frac{5}{2}x^3$

$f'(x) = -3x + \frac{15}{2}x^2$

$f''(x) = -3 + 15x$

$f'''(x) = 15$

$f'(0) = 0, \quad f''(0) = -3, \quad f'''(0) = 15$



6) $E_n(x) = \sum_{r=0}^{n} \frac{x^r}{r!}$

i)

$$E_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$E_n'(x) = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \dots + \frac{nx^{n-1}}{n!}$$

$$= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{n!}$$

$$= E_{n-1}(x)$$

ii)

$$\int E_n(x)\,dx$$

$$= \int 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}\,dx$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3 \times 2!} + \dots \frac{x^{n+1}}{(n+1)n!} + c$$

$$= 1 + x + \frac{x^2}{2!} + \dots \frac{x^{n+1}}{(n+1)!} + c$$

with 1 coming from within constant

$$= E_{n+1}(x)$$

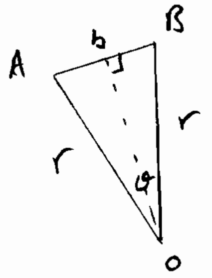
$E_\infty(x)$ is the Maclaurin expansion for e^x

$$\frac{d}{dx} e^x = e^x \qquad \int e^x\,dx = e^x + c$$

as with the work above.

7)

i) $c = \dfrac{8b - a}{3}$

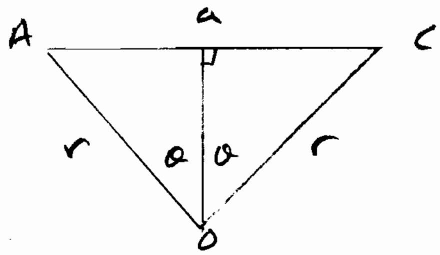


$\perp$ line from O to AB bisects AB and bisects $\angle$ AOB

$$\therefore \sin\left(\frac{\theta}{2}\right) = \frac{\frac{b}{2}}{r}$$

$$r\sin\left(\frac{\theta}{2}\right) = \frac{b}{2}$$

$$2r\sin\left(\frac{\theta}{2}\right) = b$$



Line bisecting $\angle$ AOC also bisects AC

$$\therefore \sin\theta = \frac{\frac{a}{2}}{r}$$

$$r\sin\theta = \frac{a}{2}$$

$$2r\sin\theta = a$$

ii)

$$b \approx 2r\left(\frac{\theta}{2} - \frac{\left(\frac{\theta}{2}\right)^3}{6}\right)$$

$$b \approx 2r\left(\frac{\theta}{2} - \frac{\theta^3}{48}\right)$$

$$b \approx r\left(\theta - \frac{\theta^3}{24}\right)$$

7ii) cont)

$a = 2r\sin\theta$

$a \approx 2r\left(\theta - \frac{\theta^3}{6}\right)$

$a \approx r\left(2\theta - \frac{\theta^3}{3}\right)$

$\therefore\ 8b - a \approx 8r\left(\theta - \frac{\theta^3}{24}\right) - r\left(2\theta - \frac{\theta^3}{3}\right)$

$= 8r\theta - \cancel{\frac{\theta^3}{3}} - 2r\theta + \cancel{\frac{\theta^3}{3}}$

$= 6r\theta$

$\therefore\ 8b - a \approx 6r\theta$

Now circular arc length for angle $2\theta = 2r\theta$

but $\frac{8b-a}{3} \approx \frac{6r\theta}{3} = 2r\theta$

$\therefore$ rule verified

iii)

When $\theta = \frac{\pi}{3}$

$a = 2r\sin\theta = 2r \times \frac{\sqrt{3}}{2}$

$= r\sqrt{3}$

$b = 2r\sin\frac{\theta}{2} = 2r \times \frac{1}{2} = r$

$\therefore$ using rule

$c = \frac{8b-a}{3} = \frac{8r - r\sqrt{3}}{3}$

$= 2.089316397\,r$

Actual arc length $= 2r\theta = \frac{2\pi r}{3}$

$= 2.094395102\,r$

$\%\text{ error} = \dfrac{\frac{8-\sqrt{3}}{3} - \frac{2\pi}{3}}{\frac{2\pi}{3}} \times 100$

$= \frac{8 - \sqrt{3} - 2\pi}{2\pi} \times 100$

$= -0.2425\ \%$

8)

α radians $= \frac{\alpha \times 180}{\pi}^{\circ}$

$\frac{3}{2}\left(\frac{180\alpha}{\pi}\right)^2 = 4924\alpha^2$

$\approx 5000\alpha^2$

For say $\alpha = 0.1$

Correction $= 5000 \times 0.01$ cm

$= 50$ cm

Giving length of 99.5 m

Actual length $= 100\cos\alpha$

$= 99.50$ m

For say $\alpha = 0.2$

correction $= 5000 \times 0.04 = 200$ cm

Giving length of 98 m

Actual length $= 100\cos\alpha$

$= 98.01$ m

$\therefore$ accurate for gentle slopes.

9) Let $f(x) = e^{-\frac{1}{2}x^2}$

$f'(x) = -xe^{-\frac{1}{2}x^2}$

$f''(x) = x^2e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2}$

$= (x^2-1)e^{-\frac{1}{2}x^2}$

$f^{(3)}(x) = (x^2-1)(-xe^{-\frac{1}{2}x^2}) + e^{-\frac{1}{2}x^2}(2x)$

$= (-x^3 + x + 2x)e^{-\frac{1}{2}x^2}$

$= (3x - x^3)e^{-\frac{1}{2}x^2}$

$f^{(4)}(x) = (3x-x^3)(-xe^{-\frac{1}{2}x^2}) + e^{-\frac{1}{2}x^2}(3-3x^2)$

$= e^{-\frac{1}{2}x^2}(x^4 - 3x^2 - 3x^2 + 3)$

$= (x^4 - 6x^2 + 3)e^{-\frac{1}{2}x^2}$

$f^{(5)}(x) = (x^4-6x^2+3)(-xe^{-\frac{1}{2}x^2}) + e^{-\frac{1}{2}x^2}(4x^3 - 12x)$

$= e^{-\frac{1}{2}x^2}(-x^5 + 6x^3 - 3x + 4x^3 - 12x)$

$= (-x^5 + 10x^3 - 15x)e^{-\frac{1}{2}x^2}$

$f^{(6)}(x) = (-x^5+10x^3-15x)(-xe^{-\frac{1}{2}x^2}) + e^{-\frac{1}{2}x^2}(-5x^4 + 30x^2 - 15)$

$= e^{-\frac{1}{2}x^2}[x^6 - 15x^4 + 45x^2 - 15]$

$\Rightarrow f(0) = 1$

$f'(0) = 0$

$f''(0) = -1$

$f'''(0) = 0$

$f^{(4)}(0) = 3$

$f^{(5)}(0) = 0$

$f^{(6)}(0) = -15$

$e^{-\frac{1}{2}x^2} \approx 1 - \frac{x^2}{2} + \frac{3x^4}{24} - \frac{15x^6}{720}$

$= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48}$

etc etc !!