1) i) $\sin 3x \approx 3x - \frac{(3x)^3}{3!}$

$= 3x - \frac{9x^3}{2}$

ii) $\cos 2x \approx 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}$

$= 1 - 2x^2 + \frac{2x^4}{3}$

iii) $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\approx \frac{1}{2}\left(1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!}\right)\right)$

$= \frac{1}{2}\left(\frac{4x^2}{2} - \frac{16x^4}{4!}\right)$

$= x^2 - \frac{x^4}{3}$

iii) Alternative Solution

$\sin^2 x \approx \left(x - \frac{x^3}{6}\right)^2$

$= \left(x - \frac{x^3}{6}\right)\left(x - \frac{x^3}{6}\right)$

$\approx x^2 - \frac{x^4}{6} - \frac{x^4}{6}$

$= x^2 - \frac{x^4}{3}$

iv) $\ln(1 + \sin x)$

$\approx \ln\left(1 + \left(x - \frac{x^3}{6}\right)\right)$

$\approx \left(x - \frac{x^3}{6}\right) - \frac{\left(x - \frac{x^3}{6}\right)^2}{2} + \frac{\left(x - \frac{x^3}{6}\right)^3}{3} - \frac{\left(x - \frac{x^3}{6}\right)^4}{4}$

$\approx x - \frac{x^3}{6} - \frac{\left(x^2 - \frac{x^4}{3}\right)}{2} + \frac{x^3}{3} - \frac{x^4}{4}$

$= x - \frac{x^3}{6} - \frac{x^2}{2} + \frac{x^4}{6} + \frac{x^3}{3} - \frac{x^4}{4}$

$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$

v) $e^{-x}\sin x$

$e^{-x} \approx \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}\right)$

$\sin x \approx \left(x - \frac{x^3}{6}\right)$

$e^{-x}\sin x \approx x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} - \frac{x^3}{6} + \frac{x^4}{6}$

$= x - x^2 + \frac{x^3}{3}$

vi) $e^{\sin x} \approx e^{\left(x - \frac{x^3}{6}\right)}$

$\approx 1 + \left(x - \frac{x^3}{6}\right) + \frac{\left(x - \frac{x^3}{6}\right)^2}{2}$

$+ \frac{\left(x - \frac{x^3}{6}\right)^3}{6} + \frac{\left(x - \frac{x^3}{6}\right)^4}{24}$

1vi) cont

$$\approx 1 + x - \frac{x^3}{6} + \frac{x^2}{2} - \frac{x^4}{6} + \frac{x^3}{6} + \frac{x^4}{24}$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

2)

i) $$\int \frac{1}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\sin^{-1}2x + c$$

ii) $(1-4x^2)^{-\frac{1}{2}}$

$$\approx 1 + -\tfrac{1}{2}(-4x^2) + \frac{-\frac{1}{2}\cdot-\frac{3}{2}}{1.2}(-4x^2)^2$$

$$+ \frac{-\frac{1}{2}\cdot-\frac{3}{2}\cdot-\frac{5}{2}}{1.\ 2.\ \cancel{3}}(-4x^2)^3$$

$$= 1 + 2x^2 + 6x^4 + 20x^6$$

$$\arcsin 2x \approx 2\int (1 + 2x^2 + 6x^4 + 20x^6)\,dx$$

$$= 2\left[x + \frac{2x^3}{3} + \frac{6x^5}{5} + \frac{20x^7}{7}\right] + c$$

$$= 2x + \frac{4x^3}{3} + \frac{12x^5}{5} + \frac{40x^7}{7} + c$$

3)

$$\int \frac{1}{1+x^2} = \arctan x + c$$

$$\frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$\approx 1 + -1(x^2) + \frac{-1.-2}{1.2}(x^2)^2$$

$$+ \frac{-1.-2.-3}{1.2\ 3}(x^2)^3 + \ldots$$

$$\approx 1 - x^2 + x^4 - x^6 + x^8 -$$

$$\therefore \int \frac{1}{1+x^2} \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} -$$

$$\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots$$

ii) Putting $x = 1$

$$\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - + \ldots$$

iii)

$$\arctan\tfrac{1}{2} = \frac{1}{2} - \frac{(\frac{1}{2})^3}{3} + \frac{(\frac{1}{2})^5}{5} - \frac{(\frac{1}{2})^7}{7} +$$

$$\arctan\tfrac{1}{3} = \frac{1}{3} - \frac{(\frac{1}{3})^3}{3} + \frac{(\frac{1}{3})^5}{5} - \frac{(\frac{1}{3})^7}{7} +$$

<u>This does not help!</u>

Try let $A = \arctan\frac{1}{2}$

let $B = \arctan\frac{1}{3}$

Then $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$.

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{1}{2}+\frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}}$$

$$= \frac{\frac{5}{6}}{\frac{5}{6}} = 1$$

$$\therefore A + B = \frac{\pi}{4}$$

$$\Rightarrow \frac{\pi}{4} = \arctan\left(\tfrac{1}{2}\right) + \arctan\left(\tfrac{1}{3}\right)$$

3iii b) Let $A = \arctan\frac{1}{5}$

Let $B = \arctan\frac{1}{239}$

Then $\tan A = \frac{1}{5}$, $\tan B = \frac{1}{239}$

Find $\tan(4A - B)$

First $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

$$= \frac{\frac{2}{5}}{1 - \left(\frac{1}{5}\right)^2}$$

$$= \frac{\frac{2}{5}}{\frac{24}{25}} = \frac{2}{5} \times \frac{25}{24} = \frac{5}{12}$$

$\therefore \tan 2A = 5/12$

$\tan 4A = \tan(2A + 2A)$

$$= \frac{2\tan 2A}{1 - \tan^2 2A}$$

$$= \frac{\frac{10}{12}}{1 - \frac{25}{144}}$$

$$= \frac{\frac{10}{12}}{\frac{119}{144}}$$

$$= \frac{10}{\cancel{12}} \times \frac{\cancel{144}^{12}}{119} = \frac{120}{119}$$

$\tan 4A = \dfrac{120}{119}$

Now $\tan(4A - B)$

$$= \frac{\tan 4A - \tan B}{1 + \tan 4A \tan B}$$

$$= \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}}$$

$$= \frac{\frac{28680 - 119}{28441}}{\frac{28441 + 120}{28441}}$$

$$= \frac{\frac{28561}{28441}}{\frac{28561}{28441}} = 1$$

$\therefore 4A - B = \frac{\pi}{4}$

$4\arctan\frac{1}{5} - \arctan\frac{1}{239} = \frac{\pi}{4}$

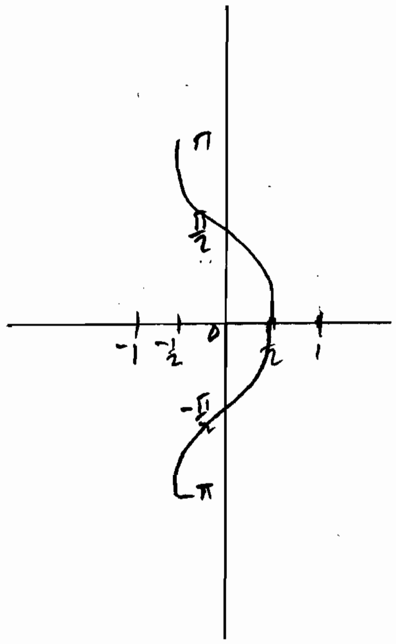
iv) $\pi = 4\left(4\arctan\frac{1}{5} - \arctan\frac{1}{239}\right)$

$= 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239}$

$\arctan\frac{1}{5} = \frac{1}{5} - \frac{\left(\frac{1}{5}\right)^3}{3} + \frac{\left(\frac{1}{5}\right)^5}{5} - \frac{\left(\frac{1}{5}\right)^7}{7}$

Too tedious!!

4) i



4 ii)

$$y = \arccos 2x$$

$$\cos y = 2x$$

$$-\sin y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = -\frac{2}{\sin y}$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{1-4x^2}}$$

---

4 iii)

$$\int \arccos(2x)\,dx$$

Let $u = \arccos 2x$

$$\frac{du}{dx} = \frac{-2}{\sqrt{1-4x^2}}$$

Let $\frac{dv}{dx} = 1$

$$v = x$$

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

$$\int \arccos(2x)\,dx$$

$$= x\arccos x + \int \frac{2x}{\sqrt{1-4x^2}}\,dx \quad (*)$$

Now $\int \frac{2x}{\sqrt{1-4x^2}}\,dx$

Let $u = 1-4x^2$

$$\frac{du}{dx} = -8x$$

$$du = -8x\,dx$$

$$-\frac{1}{4}du = 2x\,dx$$

Integral becomes

$$\int -\frac{1}{4u^{1/2}}\,du$$

$$= \frac{-\frac{1}{4}u^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}u^{\frac{1}{2}} + c$$

$$= -\frac{1}{2}\sqrt{1-4x^2} + c$$

Substituting in (*) gives answer

$$\int \arccos 2x\,dx$$

$$= x\arccos(2x) - \frac{\sqrt{1-4x^2}}{2} + c$$

4iv) $(1-4x^2)^{-\frac{1}{2}}$

$$\approx 1 + -\tfrac{1}{2}(-4x^2) + \frac{-\frac{1}{2}\cdot -\frac{3}{2}}{1.2}(-4x^2)^2$$

$$= 1 + 2x^2 + 6x^4$$

$$\arccos 2x = \int -2(1-4x^2)^{-\frac{1}{2}}\,dx$$

$$\approx -2\int (1+2x^2+6x^4)\,dx$$

$$= -2\left(x + \frac{2x^3}{3} + \frac{6x^5}{5}\right) + c$$

$$= -2x - \frac{4x^3}{3} - \frac{12x^5}{5} + c$$

Since $\arccos 0 = \frac{\pi}{2}$

$$c = \frac{\pi}{2}$$

$$\arccos 2x \approx \frac{\pi}{2} - 2x - \frac{4x^3}{3} - \frac{12x^5}{5}$$