MEI MECHANICS 2 CENTRE OF MASS EXERCISE 4B

1)

i)

2m at (1, 1)
3m at (3, 0)
m at (4, 3)
4m at (2, −2)

$$10m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 2m\begin{pmatrix}1\\1\end{pmatrix} + 3m\begin{pmatrix}3\\0\end{pmatrix} + m\begin{pmatrix}4\\3\end{pmatrix} + 4m\begin{pmatrix}2\\-2\end{pmatrix}$$

$$\begin{pmatrix}10\bar{x}\\10\bar{y}\end{pmatrix} = \begin{pmatrix}2+9+4+8\\2+0+3-8\end{pmatrix} = \begin{pmatrix}23\\-3\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}2\cdot3\\-0\cdot3\end{pmatrix}$$

ii)

m at (−3, 0)
m at (3, 0)
2m at (−2, 1)
2m at (2, 1)
3m at (−1, 2)
3m at (1, 2)
4m at (0, 3)

Symmetrical about y axis

$$\therefore \quad \bar{x} = 0$$

$$16m \times \bar{y} = 2m\times0 + 4m\times1 + 6m\times2 + 4m\times3$$

$$16\bar{y} = 0 + 4 + 12 + 12 = 28$$

$$\bar{y} = \frac{28}{16} = 1.75$$

$$(\bar{x}, \bar{y}) = (0, 1.75)$$

iii)

4m at (−3, 0)
5m at (3, 0)
5m at (−2, 0)
4m at (2, 0)
m at (0, 2)
2m at (0 3)
2m at (0, −1)
m at (0, −2)

$$24m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 4m\begin{pmatrix}-3\\0\end{pmatrix} + 5m\begin{pmatrix}3\\0\end{pmatrix} + 5m\begin{pmatrix}-2\\0\end{pmatrix} + 4m\begin{pmatrix}2\\0\end{pmatrix} + m\begin{pmatrix}0\\2\end{pmatrix} + 2m\begin{pmatrix}0\\3\end{pmatrix} + 2m\begin{pmatrix}0\\-1\end{pmatrix} + m\begin{pmatrix}0\\-2\end{pmatrix}$$

$$24\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}-12+15-10+8+0+0+0+0\\0+0+0+0+2+6-2-2\end{pmatrix}$$

$$\begin{pmatrix}24\bar{x}\\24\bar{y}\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$$

$$(\bar{x}, \bar{y}) = \left(\tfrac{1}{24}, \tfrac{1}{6}\right)$$

iv)

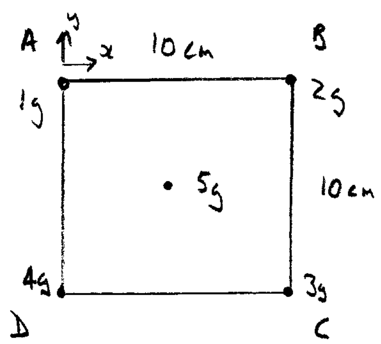
2m at (1, 1)
2m at (−1, −3)
3m at (−4, −1)
2m at (−5, −2)
m at (−5, −4)

$$10m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 2m\begin{pmatrix}1\\1\end{pmatrix} + 2m\begin{pmatrix}-1\\-3\end{pmatrix} + 3m\begin{pmatrix}-4\\-1\end{pmatrix} + 2m\begin{pmatrix}-5\\-2\end{pmatrix} + m\begin{pmatrix}-5\\-4\end{pmatrix}$$

$$10\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}2-2-12-10-5\\2-6-3-4-4\end{pmatrix} = \begin{pmatrix}-27\\-15\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}-2\cdot7\\-1\cdot5\end{pmatrix}$$

2)



2 cont) $15g\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 1g\begin{pmatrix}0\\ 0\end{pmatrix} + 2g\begin{pmatrix}10\\ 0\end{pmatrix} + 3g\begin{pmatrix}10\\ -10\end{pmatrix}$

$+ 4g\begin{pmatrix}0\\ -10\end{pmatrix} + 5g\begin{pmatrix}5\\ -5\end{pmatrix}$

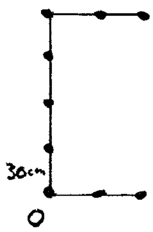
$$\begin{pmatrix}15\bar{x}\\ 15\bar{y}\end{pmatrix} = \begin{pmatrix}0+20+30+0+25\\ 0+0-30-40-25\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\frac{75}{15}\\ -\frac{95}{15}\end{pmatrix} = \begin{pmatrix}5\\ -19/3\end{pmatrix}$$

$$(\bar{x}, \bar{y}) = \left(5, -\frac{19}{3}\right)$$

3)

i)



Bulb + holder = 200g

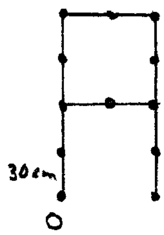
Let 200g = 1 unit of mass

$$9\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}60+30+0+0+0+0+0+30+60\\ 0+0+0+30+60+90+120+120+120\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}180/9\\ 540/9\end{pmatrix} = \begin{pmatrix}20\\ 60\end{pmatrix}$$

$$(\bar{x}, \bar{y}) = (20\text{cm}, 60\text{cm})$$

3 ii)



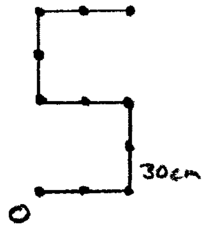
Let 200g be 1 unit of mass

$$12\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0+0+0+0+0+30+60+60+60+60+60+30\\ 0+30+60+90+120+120+120+90+60+30+0+60\end{pmatrix}$$

$$12\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}360\\ 780\end{pmatrix}$$

$$(\bar{x}, \bar{y}) = (30\text{cm}, 65\text{cm})$$

3 iii)



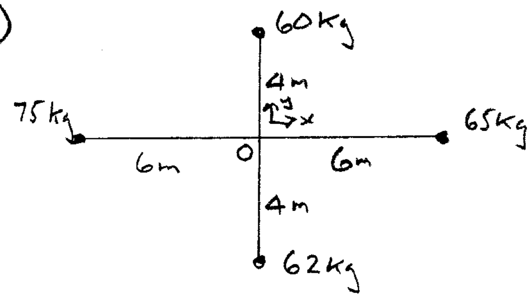
Let 200g be 1 unit of mass

$$11\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0+30+60+60+60+30+0+0+0+30+60\\ 0+0+0+30+60+60+60+90+120+120+120\end{pmatrix}$$

$$11\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}330\\ 660\end{pmatrix}$$

$$(\bar{x}, \bar{y}) = (30\text{cm}, 60\text{cm})$$

4)



$$262\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 75\begin{pmatrix}-6\\ 0\end{pmatrix} + 60\begin{pmatrix}0\\ 4\end{pmatrix} + 65\begin{pmatrix}6\\ 0\end{pmatrix} + 62\begin{pmatrix}0\\ -4\end{pmatrix}$$

$$262\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}-450+0+390+0\\ 0+240+0-248\end{pmatrix}$$

$$262\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}-60\\ -8\end{pmatrix}$$

$$(\bar{x}, \bar{y}) = \left(\frac{-60}{262}, \frac{-8}{262}\right)$$

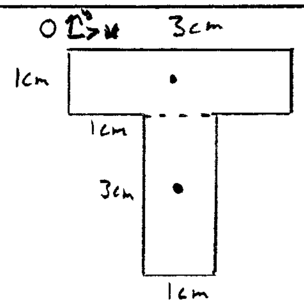
4 cont) Radius of circle around O

$$= \sqrt{\left(\frac{-60}{262}\right)^2 + \left(\frac{-8}{262}\right)^2}$$

$$= 0.231 \text{ m}$$

5)

i)

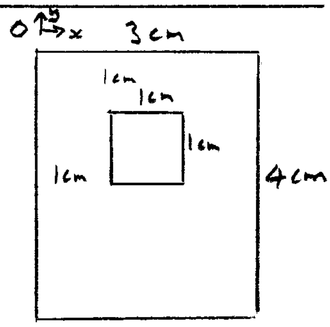


Let 1cm^2 = 1 unit of mass

$$6\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 3\begin{pmatrix}1.5\\ -0.5\end{pmatrix} + 3\begin{pmatrix}1.5\\ -2.5\end{pmatrix}$$

$$6\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}4.5 + 4.5\\ -1.5 - 7.5\end{pmatrix} = \begin{pmatrix}9\\ -9\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}1.5\\ -1.5\end{pmatrix}$$



$$11\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 12\begin{pmatrix}1.5\\ -2\end{pmatrix} - 1\begin{pmatrix}1.5\\ -1.5\end{pmatrix}$$

$$11\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}18 - 1.5\\ -24 + 1.5\end{pmatrix} = \begin{pmatrix}16.5\\ -22.5\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}1.5\\ -2.05\end{pmatrix}$$

Let z direction be vertically upwards

ii)

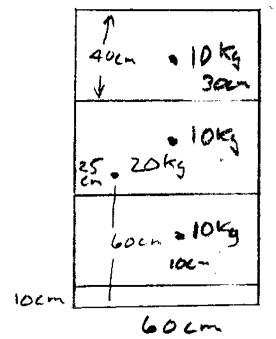
$$6\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = 2\begin{pmatrix}1\\ -0.5\\ 0\end{pmatrix} + 3\begin{pmatrix}1.5\\ -2.5\\ 0\end{pmatrix} + 1\begin{pmatrix}2\\ -0.5\\ 0.5\end{pmatrix}$$

$$6\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}2 + 4.5 + 2\\ -1 - 7.5 - 0.5\\ 0 + 0 + 0.5\end{pmatrix} = \begin{pmatrix}8.5\\ -9\\ 0.5\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}1.42\\ -1.50\\ 0.08\end{pmatrix}$$

6)

i)



$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 20\begin{pmatrix}25\\ 60\end{pmatrix} + 10\begin{pmatrix}30 + 30 + 30\\ 20 + 60 + 100\end{pmatrix}$$

$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}1400\\ 3000\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}28\\ 60\end{pmatrix}$$

ii)

$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 20\begin{pmatrix}25\\ 60\end{pmatrix} + 10\begin{pmatrix}30 + 90 + 90\\ 20 + 60 + 100\end{pmatrix}$$

$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}2600\\ 3000\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}52\\ 60\end{pmatrix}$$

6iii) All 3 drawers open

$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 20\begin{pmatrix}25\\ 60\end{pmatrix} + 10\begin{pmatrix}90+90+90\\ 20+60+100\end{pmatrix}$$

$$50\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}3200\\ 3000\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}64\\ 60\end{pmatrix}$$

Cabinet tips since $\bar{x}$ is 4cm beyond front of cabinet

6iv)

On point of tipping $\bar{x} = 60\text{cm}$

$$50 \times 60 = 20\times25 + 10\times90 + 10\times90 + 10\times q$$

$$3000 = 500 + 900 + 900 + 10q$$

$$700 = 10q$$

$$\Rightarrow q = 70\text{ cm}$$

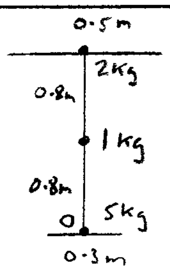
where q is distance of centre of mass of third drawer from back of cabinet.

Third drawer can be open up to

$$70 - 30 = 40\text{cm}$$

7)

i)

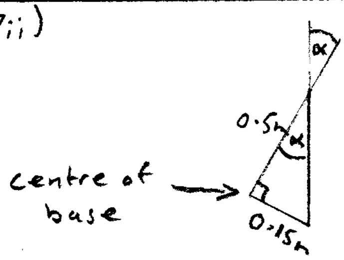


$$8\bar{x} = 5\times0 + 1\times0.8 + 2\times1.6$$

$$8\bar{x} = 4$$

$$\bar{x} = 0.5\text{ m}$$ above centre of base

7ii)

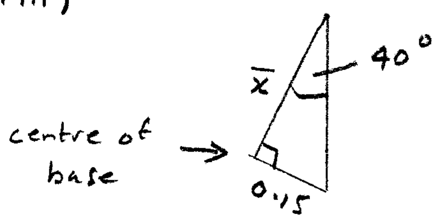


$$\alpha = \tan^{-1}\left(\frac{0.15}{0.5}\right)$$

$$\alpha = 16.7^\circ$$

Can be tipped 16.7° before toppling

7iii)



$$\tan 40^\circ = \frac{0.15}{\bar{x}}$$

$$\bar{x} = \frac{0.15}{\tan 40^\circ} = 0.179\text{m}$$

$$\approx 0.18\text{m}$$

7iv) Let mass of base be m

$$(m+3)\bar{x} = m\times0 + 1\times0.8 + 2\times1.6$$

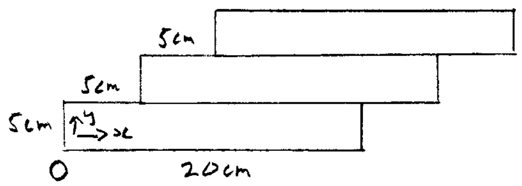
$$(m+3)0.178763 = 4$$

$$m+3 = \frac{4}{0.178763}$$

$$m = \frac{4}{0.178763} - 3$$

$$m = 19.4\text{ kg}$$

8)



i) a) 1 brick $(\bar{x}, \bar{y}) = (10, 2.5)$

Let each brick have mass of 1 unit

b) 2 bricks

$$2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}10\\ 2.5\end{pmatrix} + \begin{pmatrix}15\\ 7.5\end{pmatrix} = \begin{pmatrix}25\\ 10\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}12.5\\ 5\end{pmatrix}$$

c) 3 bricks

$$3\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}25\\ 10\end{pmatrix} + \begin{pmatrix}20\\ 12.5\end{pmatrix} = \begin{pmatrix}45\\ 22.5\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}15\\ 7.5\end{pmatrix}$$

d) 4 bricks

$$4\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}45\\ 22.5\end{pmatrix} + \begin{pmatrix}25\\ 17.5\end{pmatrix} = \begin{pmatrix}70\\ 40\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}17.5\\ 10\end{pmatrix}$$

e) 5 bricks

$$5\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}70\\ 40\end{pmatrix} + \begin{pmatrix}30\\ 22.5\end{pmatrix} = \begin{pmatrix}100\\ 62.5\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}20\\ 12.5\end{pmatrix}$$

8 ii) With 5 bricks system is on point of toppling since C of M is above right hand edge of bottom brick. A sixth brick would cause toppling.

8 iii) If displacement is 2 cm

2 bricks $\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}11\\ 5\end{pmatrix}$

3 bricks $\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}12\\ 7.5\end{pmatrix}$

11 bricks $\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}20\\ 27.5\end{pmatrix}$

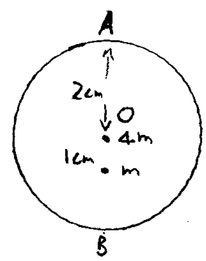
11 bricks will bring system to point of toppling. 12th brick would cause toppling.

n bricks $\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}10 + n - 1\\ 2.5 + (n-1)2.5\end{pmatrix}$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}9 + n\\ 2.5 \times n\end{pmatrix}$$

9)

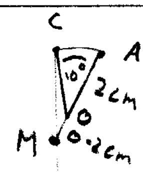
i)



$$5m\bar{x} = 4m \times 0 + m \times 1 = m$$

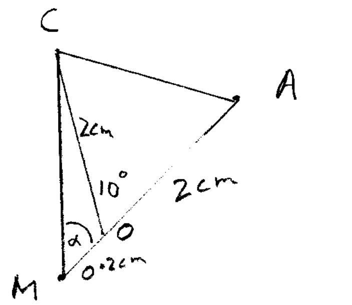
$$\bar{x} = 0.2 \text{ cm below } O$$

9 ii)



Centre of mass at M

9ii) cont



Cosine Rule in $\triangle AOC$

$$AC^2 = 2^2 + 2^2 - 2\times 2\times 2\cos 10^\circ$$

$$\Rightarrow AC = 0.348622971 \text{ cm}$$

$$\angle OAC = \frac{180-10}{2} = 85^\circ \text{ (isos } \triangle)$$

Cosine Rule in $\triangle ACM$

$$CM^2 = AM^2 + AC^2 - 2\times AM\times AC\cos 85^\circ$$

$$CM^2 = 0.348622971^2 + 2.2^2 - 2\times 0.348622971\times 2.2\cos 85^\circ$$

$$\Rightarrow CM = 2.197236037 \text{ cm}$$

Sine Rule in $\triangle ACM$

$$\frac{CM}{\sin 85^\circ} = \frac{AC}{\sin\alpha}$$

$$\Rightarrow \sin\alpha = \frac{AC\sin 85^\circ}{CM}$$

$$\Rightarrow \alpha = \sin^{-1}\left(\frac{0.348622971\sin 85^\circ}{2.197236037}\right)$$

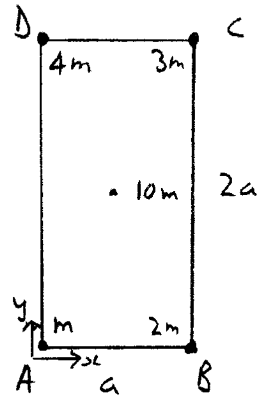
$$\alpha = 9.09^\circ$$

Angle between AB and vertical

$$\approx 9.1^\circ$$

10)

i)



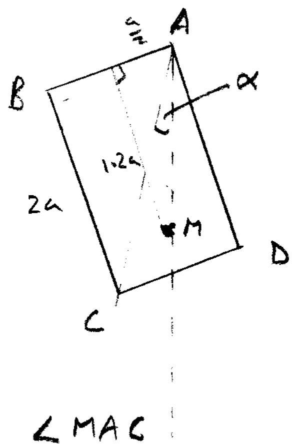
$$20m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 10m\begin{pmatrix}\frac{a}{2}\\ a\end{pmatrix} + m\begin{pmatrix}0\\ 0\end{pmatrix} + 2m\begin{pmatrix}a\\ 0\end{pmatrix} + 3m\begin{pmatrix}a\\ 2a\end{pmatrix} + 4m\begin{pmatrix}0\\ 2a\end{pmatrix}$$

$$20\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}5a + 0 + 2a + 3a + 0\\ 10a + 0 + 0 + 6a + 8a\end{pmatrix}$$

$$20\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}10a\\ 24a\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0.5a\\ 1.2a\end{pmatrix}$$

10 ii)



Find $\angle MAC$

$$\angle MAC = \angle MAB - \angle CAB$$

$$= \tan^{-1}\left(\frac{1.2a}{0.5a}\right) - \tan^{-1}\left(\frac{2a}{a}\right)$$

$$\angle MAC = 3.945^\circ \approx 3.9^\circ$$

10 iii) Require centre of mass to lie on AC

$\therefore$ need $\bar{x} = \frac{1}{2}\bar{y}$

Let mass at D be M

$$(16m+M)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} =$$

$$10m\begin{pmatrix}\frac{a}{2}\\ a\end{pmatrix} + 2m\begin{pmatrix}a\\ 0\end{pmatrix} + 3m\begin{pmatrix}a\\ 2a\end{pmatrix} + M\begin{pmatrix}0\\ 2a\end{pmatrix}$$

$$(16m+M)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}5ma + 2ma + 3ma + 0\\ 10ma + 0 + 6ma + 2Ma\end{pmatrix}$$

$$(16m+M)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}10ma\\ 16ma + 2Ma\end{pmatrix}$$

We require $10ma = \frac{1}{2}(16ma + 2Ma)$

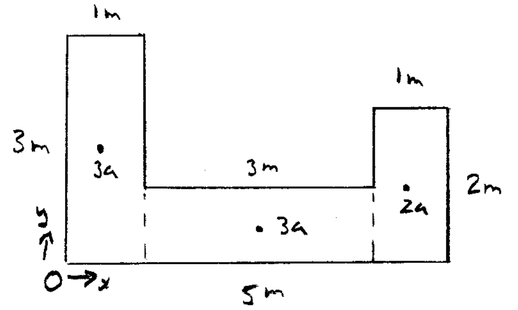
$$10ma = 8ma + Ma$$

$$2ma = Ma$$

$$\Rightarrow M = 2m$$

Mass at D should be altered to 2m

11) i)



Let mass per m^2 be a

$$8a\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 3a\begin{pmatrix}0.5\\ 1.5\end{pmatrix} + 3a\begin{pmatrix}2.5\\ 0.5\end{pmatrix} + 2a\begin{pmatrix}4.5\\ 1\end{pmatrix}$$

$$8\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}1.5 + 7.5 + 9\\ 4.5 + 1.5 + 2\end{pmatrix} = \begin{pmatrix}18\\ 8\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}18/8\\ 8/8\end{pmatrix} = \begin{pmatrix}9/4\\ 1\end{pmatrix}$$

11 ii) Suppose mass M is removed

We require $\bar{x} = 2.5$

$$(8a - M)\bar{x} = 18a - M \times 0.5$$

$$(8a - M) \times 2.5 = 18a - 0.5M$$

$$20a - 2.5M = 18a - 0.5M$$

$$2a = 2M$$

$$\Rightarrow M = a$$

$\therefore$ area to be removed $= 1\,m^2$

$$\Rightarrow \pi r^2 = 1$$

$$r = \sqrt{\frac{1}{\pi}}$$

$$r = 0.564\ m$$

Not feasible since proposed centre of circle is 0.5 m from edges.

11 iii) Let holes of radius r

be centred on $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, k)$

with each removing a mass M

11 iii) cont)

$$(8a - 2M)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$= \begin{pmatrix}18a\\ 8a\end{pmatrix} - M\begin{pmatrix}\frac{1}{2} + \frac{1}{2}\\ \frac{1}{2} + k\end{pmatrix}$$

But $P = (2.5, 1)$ so

$$(8a-2M)\begin{pmatrix}2\cdot 5\\ 1\end{pmatrix} = \begin{pmatrix}18a\\ 8a\end{pmatrix} - M\begin{pmatrix}1\\ k+0\cdot 5\end{pmatrix}$$

$$\begin{matrix}(1)\\(2)\end{matrix}\quad \begin{pmatrix}20a - 5M\\ 8a - 2M\end{pmatrix} = \begin{pmatrix}18a - M\\ 8a - Mk - 0\cdot 5M\end{pmatrix}$$

From (1)

$$2a = 4M$$

$$\Rightarrow M = \frac{a}{2}$$

Subst in (2)

$$8a - 2\left(\frac{a}{2}\right) = 8a - \left(\frac{a}{2}\right)k - \frac{1}{2}\left(\frac{a}{2}\right)$$

$$\cancel{8a} - a = \cancel{8a} - \frac{ka}{2} - \frac{a}{4}$$

$$-a = -\left(\frac{2ka + a}{4}\right)$$

$$-a = -\left(\frac{a(2k+1)}{4}\right)$$

$$\Rightarrow \frac{2k+1}{4} = 1$$

$$\Rightarrow 2k+1 = 4$$

$$\Rightarrow k = \frac{3}{2}$$

$\therefore$ centre of second hole is at $\left(\frac{1}{2}, \frac{3}{2}\right)$

Finally, find radius r of holes

Each hole removes mass $M = \frac{a}{2}$

$\therefore$ each has area $0.5\ m^2$

$$\pi r^2 = 0.5$$

$$r^2 = \frac{0.5}{\pi}$$

$$r = \sqrt{\frac{0\cdot 5}{\pi}}$$

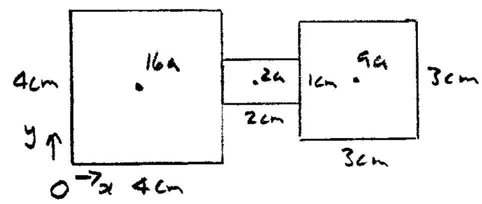
$$r = 0.399\ m$$

$$r \approx 0.40\ m$$

---

12)

i)



Let mass per cm^2 be a

$$27a\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 16a\begin{pmatrix}2\\ 2\end{pmatrix} + 2a\begin{pmatrix}5\\ 2\end{pmatrix} + 9a\begin{pmatrix}7\cdot 5\\ 2\end{pmatrix}$$

$$27\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}32 + 10 + 67\cdot 5\\ 32 + 4 + 18\end{pmatrix} = \begin{pmatrix}109\cdot 5\\ 54\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}4\cdot 06\\ 2\end{pmatrix}$$

---

12 ii)

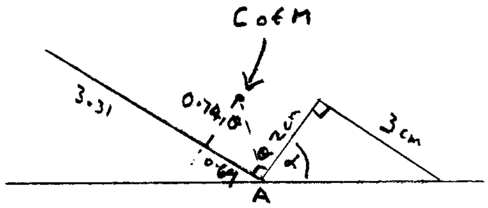
$$27\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = 16\begin{pmatrix}2\\ 2\\ 0\end{pmatrix} + 2\begin{pmatrix}4\\ 2\\ 1\end{pmatrix} + 9\begin{pmatrix}5\cdot 5\\ 2\\ 2\end{pmatrix}$$

$$27\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}32 + 8 + 49.5\\ 32 + 4 + 18\\ 0 + 2 + 18\end{pmatrix} = \begin{pmatrix}89.5\\ 54\\ 20\end{pmatrix}$$

12ii) cont)

$$\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}3.31\\ 2\\ 0.74\end{pmatrix}$$

12iii)



$$\alpha = \tan^{-1}\left(\frac{3}{2}\right) = 56.3^\circ$$

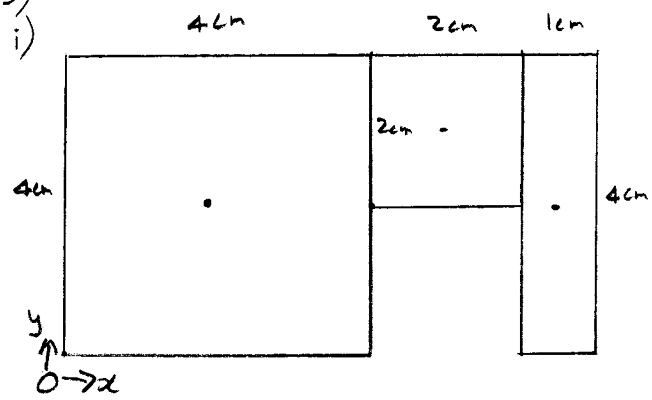
$$\theta = \tan^{-1}\left(\frac{0.69}{0.74}\right) = 43.0^\circ$$

$$\alpha + \theta = 99.3^\circ$$

$\Rightarrow$ C of M is to the left of A

$\therefore$ lamina will topple ant-clockwise about point A in diagram

13) i)



Let mass per cm^2 be a

$$24a\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 16a\begin{pmatrix}2\\ 2\end{pmatrix} + 4a\begin{pmatrix}5\\ 3\end{pmatrix} + 4a\begin{pmatrix}6.5\\ 2\end{pmatrix}$$

$$24\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}32+20+26\\ 32+12+8\end{pmatrix} = \begin{pmatrix}78\\ 52\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}3.25\\ 2.17\end{pmatrix}$$

13ii)

$$24\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = 16\begin{pmatrix}2\\ 2\\ 0\end{pmatrix} + 4\begin{pmatrix}5\\ 3\\ 0\end{pmatrix} + 4\begin{pmatrix}6\\ 2\\ 0.5\end{pmatrix}$$

$$24\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}32+20+24\\ 32+12+8\\ 0+0+2\end{pmatrix} = \begin{pmatrix}76\\ 52\\ 2\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}3.17\\ 2.17\\ 0.08\end{pmatrix}$$

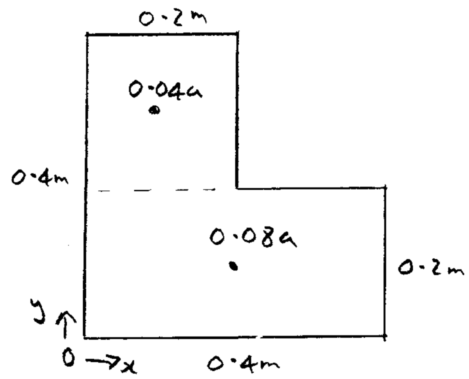
13iii)

$$24\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = 16\begin{pmatrix}2\\ 2\\ 0\end{pmatrix} + 4\begin{pmatrix}4\\ 3\\ 1\end{pmatrix} + 4\begin{pmatrix}3.5\\ 2\\ 2\end{pmatrix}$$

$$24\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}32+16+14\\ 32+12+8\\ 0+4+8\end{pmatrix} = \begin{pmatrix}62\\ 52\\ 12\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}2.58\\ 2.17\\ 0.5\end{pmatrix}$$

14) i)



Let mass per m^2 be a

$$0.12a\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 0.08a\begin{pmatrix}0.2\\ 0.1\end{pmatrix} + 0.04a\begin{pmatrix}0.1\\ 0.3\end{pmatrix}$$

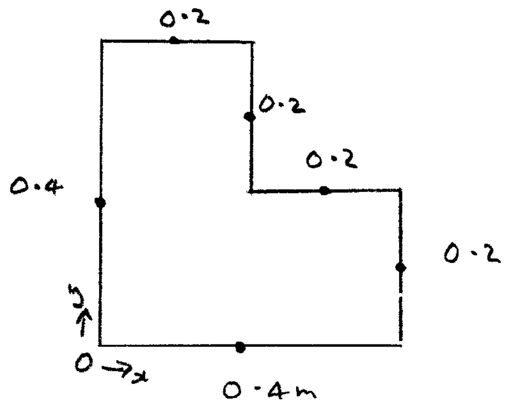
$$12\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 8\begin{pmatrix}0.2\\ 0.1\end{pmatrix} + 4\begin{pmatrix}0.1\\ 0.3\end{pmatrix}$$

14i) cont

$$12\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}1.6 + 0.4\\ 0.8 + 1.2\end{pmatrix} = \begin{pmatrix}2.0\\ 2.0\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}1/6\\ 1/6\end{pmatrix} = \begin{pmatrix}0.17\\ 0.17\end{pmatrix}$$

14ii)



Let mass per 0.1 m be a

$$16a\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 4a\begin{pmatrix}0.2\\ 0\end{pmatrix} + 2a\begin{pmatrix}0.4\\ 0.1\end{pmatrix}$$

$$+ 2a\begin{pmatrix}0.3\\ 0.2\end{pmatrix} + 2a\begin{pmatrix}0.2\\ 0.3\end{pmatrix} + 2a\begin{pmatrix}0.1\\ 0.4\end{pmatrix}$$

$$+ 4a\begin{pmatrix}0\\ 0.2\end{pmatrix}$$

$$16\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0.8 + 0.8 + 0.6 + 0.4\\ 0 + 0.2 + 0.4 + 0.6\end{pmatrix}$$

$$\begin{pmatrix}+ 0.2 + 0\\ + 0.8 + 0.8\end{pmatrix} = \begin{pmatrix}2.8\\ 2.8\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}\frac{2.8}{16}\\ \frac{2.8}{16}\end{pmatrix} = \begin{pmatrix}0.175\\ 0.175\end{pmatrix}$$

14iii)

By symmetry $\bar{z} = 0.3$ m

Let mass per m^2 of lamina be m

Area of one end $= 0.4^2 - 0.2^2$

$= 0.12\ m^2$

Area of all faces joining ends

$= 0.6 \times (0.4 + 0.2 + 0.2 + 0.2 + 0.2 + 0.4)$

$= 0.96\ m^2$

Mass of both ends $= 0.24m$
Mass of other faces $= 0.96m$

Other faces can be treated as the rods in part (ii)

$$(0.96 + 0.24)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 0.24\begin{pmatrix}\frac{1}{6}\\ \frac{1}{6}\end{pmatrix} + 0.96\begin{pmatrix}0.175\\ 0.175\end{pmatrix}$$

$$1.2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0.208\\ 0.208\end{pmatrix}$$

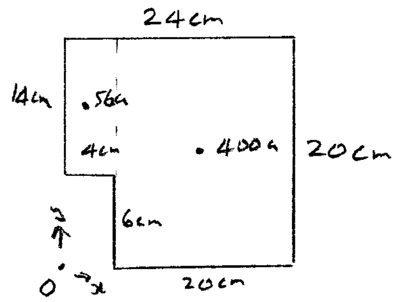
$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0.173\\ 0.173\end{pmatrix}$$

Solution $\begin{pmatrix}\bar{x}\\ \bar{y}\\ \bar{z}\end{pmatrix} = \begin{pmatrix}0.173\\ 0.173\\ 0.3\end{pmatrix}$

with all units in metres

15)

i)



Let mass per cm^2 be a

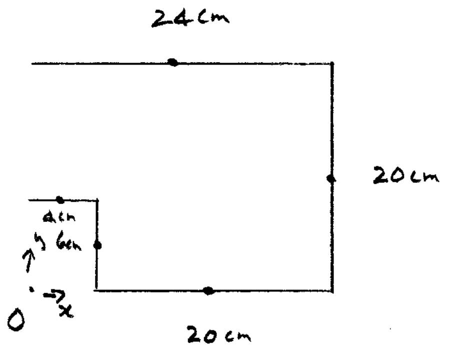
$$456a\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 56a\begin{pmatrix}2\\ 13\end{pmatrix} + 400a\begin{pmatrix}14\\ 10\end{pmatrix}$$

15 i cont)

$$456\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}112 + 5600\\ 728 + 4000\end{pmatrix} = \begin{pmatrix}5712\\ 4728\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}12.5\\ 10.4\end{pmatrix}$$

15ii)



Let mass per cm be a

$$74a\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 4a\begin{pmatrix}2\\ 6\end{pmatrix} + 6a\begin{pmatrix}4\\ 3\end{pmatrix} + 20a\begin{pmatrix}14\\ 0\end{pmatrix}$$

$$+ 20a\begin{pmatrix}24\\ 10\end{pmatrix} + 24a\begin{pmatrix}12\\ 20\end{pmatrix}$$

$$74\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}8 + 24 + 280 + 480 + 288\\ 24 + 18 + 0 + 200 + 480\end{pmatrix}$$

$$74\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}1080\\ 722\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}14.6\\ 9.8\end{pmatrix}$$

15iii)

In cross-section the faces joining the ends can be considered as the rods in part (ii)

If mass per cm^2 = 'a' on end faces, then a rod representing a face joining the ends would have mass per cm = 60a

To find $\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$ consider two end faces and the system of 5 rods

$$(2\times 456 + 74\times 60)a\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

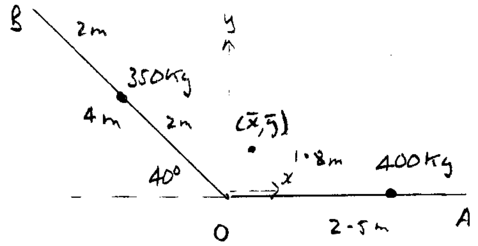
$$= (2\times 456)a\begin{pmatrix}12.5\\ 10.4\end{pmatrix} + (74\times 60)a\begin{pmatrix}14.6\\ 9.8\end{pmatrix}$$

$$5352\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}11400 + 64824\\ 9484.8 + 43512\end{pmatrix}$$

$$5352\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}76224\\ 52996.8\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}14.2\\ 9.9\end{pmatrix}$$

16) i)



$$750\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 400\begin{pmatrix}1.8\\ 0\end{pmatrix} + 350\begin{pmatrix}-2\cos 40\\ 2\sin 40\end{pmatrix}$$

$$750\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}183.769\\ 449.951\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}0.245\\ 0.600\end{pmatrix}$$

ii)

Distance of $(\bar{x},\bar{y})$ from O

$$= \sqrt{0.245^2 + 0.6^2}$$

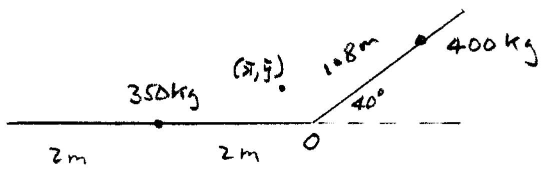
$$= 0.648\text{ m}$$

16 ii) cont) As bridge is opened, the centre of mass will move along the arc of a circle centre O radius 0.648m

16iii)

We have already seen that when the bridge is closed $\bar{x} = 0.245$m so centre of mass is to the right of the pivot point O. The bridge cannot rotate any further clockwise so its position is stable.

Now consider bridge in open position



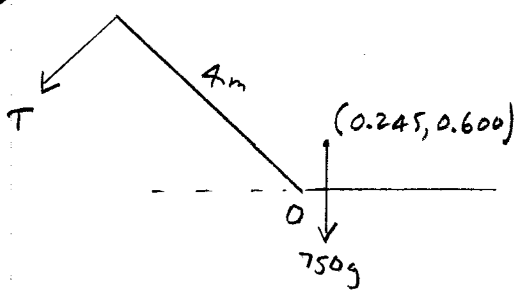
$$750\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = 350\begin{pmatrix}-2\\ 0\end{pmatrix} + 400\begin{pmatrix}1.8\cos 40\\ 1.8\sin 40\end{pmatrix}$$

$$750\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}-148.448\\ 462.807\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \begin{pmatrix}-0.198\\ 0.617\end{pmatrix}$$

Since $\bar{x} < 0$, centre of mass is to the left of pivot point O. The bridge cannot rotate any further anti-clockwise so again its position is stable.

16 iv)



Moments about O

At point of starting to turn

$$4 \times T = 0.245 \times 750g$$

$$T = \frac{0.245 \times 750g}{4}$$

$$T = 450 \text{ N}$$

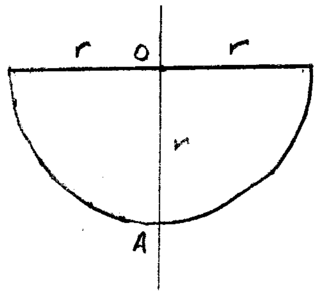
17)

i) Area of open hemisphere $= \frac{4\pi r^2}{2} = 2\pi r^2$

Area of circular disc $= \pi r^2$

Let disc have mass m

$\Rightarrow$ open hemisphere has mass 2m



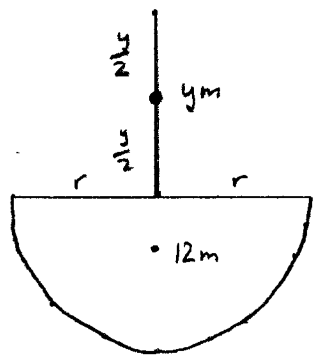
Due to symmetry of both parts, centre of mass lies on OA

$$3m\bar{x} = m \times 0 + 2m \times \frac{r}{2}$$

$$3m\bar{x} = mr$$

$$\bar{x} = \frac{r}{3}$$

17ii)



17ii) cont) Taking positive direction to be upwards from O away from hemisphere

$$(12+y)m\,\bar{x} = ym \times \frac{y}{2} - 12m \times \frac{r}{3}$$

$$(12+y)\,\bar{x} = \frac{y^2}{2} - 4r$$

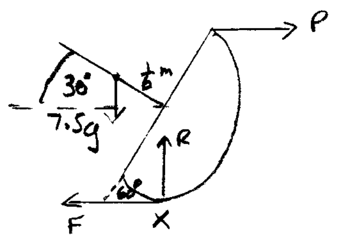
$$\bar{x} = \frac{\frac{y^2 - 8r}{2}}{(12+y)}$$

$$\bar{x} = \frac{y^2 - 8r}{24 + 2y}$$

away from hemisphere

17iii) When $m = 0.5$, $y = 3$, $r = 0.5$

$$\bar{x} = \frac{3^2 - 8 \times 0.5}{24 + 2 \times 3} = \frac{1}{6}\text{ m}$$



Total mass $= (12+y)m = 15 \times 0.5$

$= 7.5$ kg

Taking moments about X (point of contact with table)

$$7.5g \times \tfrac{1}{6}\cos 30° = P(r + r\sin 60°)$$

$$7.5g \times \tfrac{1}{6}\cos 30° = P(0.5 + 0.5\sin 60°)$$

$$P = \frac{7.5 \times 9.8 \times \frac{1}{6}\cos 30°}{(0.5 + 0.5\sin 60°)}$$

$$P = 11.4\text{ N}$$

Resolving horizontally

$$F = P$$

$$F = 11.37\text{ N}$$

On point of slipping

$$F = \mu R \quad = \mu \times 7.5g$$

$$\Rightarrow \quad 11.37 = \mu \times 7.5g$$

$$\mu = \frac{11.37}{7.5 \times 9.8}$$

$$\mu = 0.155$$

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